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SOME IMPROVEMENTS OF THE ORDER OF THE CONVERGENCE OF FINITE VOLUME SOLUTIONS

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Abstract. In this article, we improve the order of the convergence of some finite volume solutions approximating some second order elliptic problems.
In one dimensional space, we prove that finite volume approximations of order $O(h^{k+1})$, with k integer, can be obtained after k correction using the same scheme of three points and changing only the second members of the original system.
This is done for general smooth second order elliptic problems. These results can be extended for non linear second order equation $u'' = f(x, u, u')$ where f is a smooth function .
In two dimensional space, we prove that finite volume approximation of order $O(h^2)$ can be obtained, starting with finite volume solution of order $O(h)$, by using the same matrix and changing only the second member of the original system.
This is done for second order elliptic problems of the form $-\Delta u + pu = f$, with Dirichlet condition.
These results can be extended to obtain finite volume approximation of order $O(h^{k+1})$.
Heart idea behind these results is the one of Fox's difference correction in the context of finite difference method.

Key words: Second elliptic boundary problems, Finite volume solution, Scheme of three points in one dimensional space, Scheme of five points in two dimensional space, Non-Uniform mesh, Higher order of convergence
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1. Introduction

Numerical methods for partial differential equations can be divided into three general categories: finite difference methods, finite element methods, finite volume methods.
Finite difference and finite element methods have been attracted much more attention than finite volume methods, consequently there is a well developed literature in finite difference/finite element methods which treats several methods for improving the order of the convergence of the approximate solutions those using lower scheme.
The desire to use low order scheme to produce highly accurate approximation in finite difference methods led Fox [9] to introduce his difference correction technique. His idea has been modified by Pereyra and Lindberg's deferred correction. Theirs ideas have been developed by many authors like Zadunaisky's global and Frank's local defect correction (for more informations see [3] and [14]). Almost, the theoretical justifications of these methods are based on the existence of a smooth asymptotic error expansion for the base scheme. The uniformity of the mesh and the contraction property have been the main tools to prove such existence of the error expansions.

In finite Element methods, defect correction technique has been used to produce highly order of convergence by using linear /bilinear finite element method. This has been introduced by Barrett et al. [2] and Moore [12] in one dimensional space by using uniform mesh and recently by using the so-called supraconvergent mesh condition in [4].
In two dimensional space, under the uniform mesh Chibi [6] (see also the idea of contraction property in this context in Gao et al. [10]) has proved that, we can do only one correction on the rectangle and corrections we wish for periodic problems (for a theoretical framework, you can also see the communication of Hackbusch in [1], pages 89-113).
In finite volume methods, the desire to improve the order of the convergence using low order scheme has not attracted the attention it merits (see the introduction of [5]). In this context, we can mention the work of Martin et al. [11], where they used defect correction method, and under uniform mesh to propose an implicit scheme that is second order accurate both in time and space and uses only first jacobian for some unsteady problems.
The aim of this article is to develop some techniques allowing us to improve the order of the convergence of the finite volume solutions on arbitrary mesh conditions for second order elliptic problems in one and two dimensional spaces.
We prove that, starting with a finite volume solution u^h of order $O(h)$ in H^1- norm, we can obtain finite volume solution of order $O(h^2)$ in H_0^1-norm, by using the same matrix that used to compute the solution u^h.
The heart idea used in this article is the fameous Fox's difference correction in the context of finite difference methods.
The order of the convergence of the finite volume solutions, on lower schemes, depends on the second derivatives of the unknown solution u .
In one dimensional space, the second derivative of u can be expanded as a combination of the solution itself, its first derivative and a given data. We use this idea to obtain an optimal approximation to the second derivative by using the values of the basic finite volume solution u^h. This approximation allowing us to correct u^h and to obtain a new approximation can be computed by the same matrix that used to compute u^h, called first correction, of order $O(h^2)$.
Other variant to compute an optimal approximation to the second derivative of u is to use the fact that is satisfying the same equation that is satisfying by the solution itself but for different second member and boundary conditions (this holds for some second order elliptic problems). This allowing to obtain an optimal approximation to the second derivative, by using always the same matrix that used to compute u^h.
We can repeat this process, successively, to obtain finite volume approximation s of orders $O(h^{k+1})$, where k is integer by using the same matrix of original system.
In two dimensional space, we use the second variant that used in one dimensional space. For the Laplacian model, the second derivatives of the unknown solution satisfy the same equation that is satisfying by the solution itself, and by the same trick that used in one dimensional space, we can obtain a new approximation to the unknown solution of order $O(h^2)$.
These resultes can be extended for some Dirichlet models $-\Delta u + pu = f$ and to obtain corrections of arbitrary order we wish. Some numerical tests justifying our theoretical results are done, too.

2. In One Dimentional case

2.1. **Basic Results and Preliminaries.** The results of this article are presented in the context of classical functions space. We denote by $C^m(\bar{\Omega})$ (Ω in our paper is either an interval in $\mathbb{R}$ or rectangle in $\mathbb{R}^2$) the space of continuous functions which together with their derivatives up to order m inclusive are in $C(\bar{\Omega})$. The norm is

$$\|w\|_{m,\infty,\bar{\Omega}} = \max_{|\alpha|\leq m}(\max_{\bar{\Omega}}|D^{\alpha}w|)$$

In all that follows the letter c stands for a generic, positive number, different at each appearance but 'constant' in that is independent of discretsiation parameter τ, i, j

Remark 2.1. *To show that the improvement order, will be presented (in one and two dimensional spaces), hold for an arbitrary admossible mesh, we try to bound each expansion with respect to* $h_{i+\frac{1}{2}}, h_i, \ldots$ *(and we do so for two dimensional space).*

Basic results given here are done in Eymard et al. [7]. Let f be a given function defind on $(0,1)$ and consider the following equation

$$\begin{cases} -u_{xx} + \alpha u_x + \beta u = f(x), \ x \in I = (0,1), \\ \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Let τ be an admissible mesh in the sens of [7], i.e. given by family $(K_i)_{i=1,\ldots,N}$, $N \in \mathbb{N}^*$, such that $K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and a family $(x_i)_{i=0,\ldots,N+1}$ such that

$$x_0 = x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{3}{2}} < \ldots < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < \ldots < x_N < x_{N+\frac{1}{2}} = x_{N+1} = 1,$$

and

$$h_i = \mathrm{m}\ (K_i) = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \text{ for } i \in \{1, \ldots, N\},$$
$$h_i^- = x_i - x_{i-\frac{1}{2}}, h_i^+ = x_{i+\frac{1}{2}} - x_i, \text{ for } i \in \{1,\ldots,N\},\ h_0^+ = h_{N+1}^- = 0,$$
$$h_{i+\frac{1}{2}} = x_{i+1} - x_i, i = 0, \ldots, N, \text{ and size}(\tau) = h = \max\{h_i, i = 1, \ldots, N\}.$$

The system to be solved for the three points scheme by finite volume method is

$$\begin{cases} Au^h = b, \\ u_0 = u_{N+1} = 0, \end{cases} \tag{2}$$

where $u^h = (u_1, \ldots, u_N)^t, b = (b_1, \ldots, b_N)^t$ with A and b are defined by

$$(Au^h)_i = \frac{1}{h_i}\left(-\frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} + \frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} + \alpha(u_i - u_{i-1})\right) + \beta u_i, \ i = 1, \ldots, N, \tag{3}$$

$$b_i = \frac{1}{h_i}\int_{K_i} f(x)\, dx, \ i = 1, \ldots, N. \tag{4}$$

The following theorem (see [7]) gives the order of the convergence of the finite volume solution of the scheme of three points (2).

Theorem 2.1. *Let* $f \in C([0,1])$ *and let* $u \in C^2([0,1])$ *be the unique solution of 1. Let* τ *be an admissible mesh. Then there exists a unique solution* u^h *of 2 and the error is bounded by*

$$\left(\sum_{0\leq i\leq N} \frac{(e_{i+1} - e_i)^2}{h_{i+\frac{1}{2}}}\right)^{\frac{1}{2}} \leq ch\|u_{2x}\|_{\infty,\bar{I}}, \tag{5}$$

$$|e_i| \leq ch\|u_{2x}\|_{\infty}, \forall i \in \{1, \ldots, N\}, \tag{6}$$

where $e_0 = e_{N+1} = 0$ *and* $e_i = u(x_i) - u_i$, *for all* $i \in \{1, \ldots, N\}$.

Remark 2.2. *The uniform estimation (6) yields that the order of the convergence in L^2 norm is at least $O(h)$, but numerical results shows that in general the order is $O(h^2)$ when $\alpha = \beta = 0$, this means that*

$$\left(\sum_{i=1}^{N} h_i e_i^2\right)^{\frac{1}{2}} \leq ch^2. \tag{7}$$

Before we will be able to give general formulation of an arbitrary correction, we present at first the first correction and after we give the second one, where additional tools will be used. The general formulation of corrections can be given later, by using the ideas of first and second correction.

2.2. **The First Correction.** By integrating both sides of equation 1 over each finite volume K_i, we get

$$-u_x(x_{i+\frac{1}{2}}) + u_x(x_{i-\frac{1}{2}}) + \alpha u(x_{i+\frac{1}{2}}) - \alpha u(x_{i-\frac{1}{2}}) + \beta \int_{K_i} u\, dx = \int_{K_i} f\, dx, \tag{8}$$

Then, the order of convergence of finite volume solution depends on the : approximation of the flux, values $u(x_{i+\frac{1}{2}})$, $u(x_{i-\frac{1}{2}})$ and the integral $\int_{K_i} u\, dx$.

We shall use this idea combined with one of Fox to improve the order of the convergence of the basic solution u^h on the same scheme, i.e. using the same matrix A that used to compute the basic solution and changing only the r.h.s of 2.

Looking now for an expansion to the error. Assuming $u \in C^3(\bar{I})$ and using Taylor's formula, we can get

$$\begin{aligned} -\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}} + \frac{u(x_i)-u(x_{i-1})}{h_{i-\frac{1}{2}}} = \;& -u_x(x_{i+\frac{1}{2}}) + u_x(x_{i-\frac{1}{2}}) - \frac{h_{i+1}^- - h_i^+}{2} u_{2x}(x_{i+\frac{1}{2}}) + \frac{h_i^- - h_{i-1}^+}{2} u_{2x}(x_{i-\frac{1}{2}}) \\ & -R^1_{i+\frac{1}{2}} + R^1_{i-\frac{1}{2}},\ \forall i \in \{1, ..., N\}. \end{aligned} \tag{9}$$

where the following estimate holds

$$|R^1_{i+\frac{1}{2}}| \leq c{h_{i+\frac{1}{2}}}^2 |u_{3x}|_\infty,\ \forall i \in \{0, ..., N\}. \tag{10}$$

On the other hand

$$u(x_{i+\frac{1}{2}}) = u(x_i) + h_i^+ u_x + S_i^1,\ \forall i \in \{0, ..., N\}, \tag{11}$$

where

$$|S_i^1| \leq c{h_i^+}^2 |u_{2x}|_\infty,\ \forall i \in \{0, ..., N\}. \tag{12}$$

Also

$$\int_{K_i} u(x)dx = h_i u(x_i) + \frac{{h_i^+}^2}{2} u_x(x_i) - \frac{{h_i^-}^2}{2} u_x(x_{i-1}) + T_i^1,\ \forall i \in \{1, ..., N\}, \tag{13}$$

where

$$|T_i^1| \leq ch^2 h_i \|u_{2x}\|_\infty, \forall i \in \{1, ..., N\}. \tag{14}$$

Substituting terms in (8) by their expansions found in the equalities (9),(11) and (13), we can obtain, $\forall i \in \{1, ..., N\}$

$$
\begin{aligned}
-\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}} &+ \frac{u(x_i)-u(x_{i-1})}{h_{i-\frac{1}{2}}} + \alpha u(x_i) - \alpha u(x_{i-1}) + \beta h_i u(x_i) = \int_{K_i} f dx \\
&- \frac{h_{i+1}^- - h_i^+}{2} u_{2x}(x_{i+\frac{1}{2}}) + \frac{h_i^- - h_{i-1}^+}{2} u_{2x}(x_{i-\frac{1}{2}}) \\
&- \alpha h_i^+ u_x(x_i) + \alpha h_{i-1}^+ u_x(x_{i-1}) - \frac{\beta}{2}\left({h_i^+}^2 u_x(x_i) - {h_i^-}^2 u_x(x_{i-1})\right) \\
&- R^1_{i+\frac{1}{2}} + R^1_{i-\frac{1}{2}} - \alpha S_i^1 + \alpha S_{i-1}^1 - \beta T_i^1.
\end{aligned} \tag{15}
$$

By Substituting $u_{2x} = \alpha u_x + \beta u - f$ in the equation (15) we get

$$
\begin{aligned}
-\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}} &+ \frac{u(x_i)-u(x_{i-1})}{h_{i-\frac{1}{2}}} + \alpha u(x_i) - \alpha u(x_{i-1}) + \beta h_i u(x_i) = \int_{K_i} f dx \\
&- \frac{h_{i+1}^- - h_i^+}{2}\left(\alpha u_x(x_{i+\frac{1}{2}}) + \beta u(x_{i+\frac{1}{2}}) - f(x_{i+\frac{1}{2}})\right) \\
&+ \frac{h_i^- - h_{i-1}^+}{2}\left(\alpha u_x(x_{i-\frac{1}{2}}) + \beta u(x_{i-\frac{1}{2}}) - f(x_{i-\frac{1}{2}})\right) \\
&- \alpha h_i^+ u_x(x_i) + \alpha h_{i-1}^+ u_x(x_{i-1}) - \frac{\beta}{2}\left((h_i^+)^2 u_x(x_i) - (h_i^-)^2 u_x(x_{i-1})\right) \\
&- R^1_{i+\frac{1}{2}} + R^1_{i-\frac{1}{2}} - \alpha S_i^1 + \alpha S_{i-1}^1 - \beta T_i^1.
\end{aligned} \tag{16}
$$

After having found an appropriate expansion of the error, we can correct the basic solution by approximating values and pointwise derivatives of the unknown solution u in this expansion, by theirs corresponding values and partial values (forward approximation) of the basic solution u^h.

The new solution $u_1^h = (u_i^1)_{i=1}^N$, called correction, obtained after these changes, will be defined on the same scheme, i.e. using the same matrix A that used to compute the basic solution u^h, i.e. $u_0^1 = u_{N+1}^1 = 0$ and $\forall i \in \{1, ..., N\}$, we have

$$
\begin{aligned}
-\frac{u_{i+1}^1 - u_i^1}{h_{i+\frac{1}{2}}} &+ \frac{u_i^1 - u_{i-1}^1}{h_{i-\frac{1}{2}}} + \alpha(u_i^1 - u_{i-1}^1) + \beta h_i u_i^1 = \int_{K_i} f dx \\
&- \frac{h_{i+1}^- - h_i^+}{2}\left(\alpha \partial_1 u_i + \beta u_i - f(x_{i+\frac{1}{2}})\right) + \frac{h_i^- - h_{i-1}^+}{2}\left(\alpha \partial_1 u_{i-1} + \beta u_{i-1} - f(x_{i-\frac{1}{2}})\right) \\
&- \alpha h_i^+ \partial_1 u_i + \alpha h_{i-1}^+ \partial_1 u_{i-1} - \frac{\beta}{2}\left((h_i^+)^2 \partial_1 u_i - (h_i^-)^2 \partial_1 u_{i-1}\right),
\end{aligned} \tag{17}
$$

where $\partial_1 u_i = \frac{u_{i+1}-u_i}{h_{i+\frac{1}{2}}}$.

2.3. **The Convergence Order of the First Correction.** To analyse the convergence of the first correction, we follow the same proof that used for proving the order of the convergence of the basic solution. By subtracting (16) from (17) side by side, the error $e_i^1 = u_i^1 - u(x_i)$ will be satisfied

$$
-\frac{e_{i+1}^1 - e_i^1}{h_{i+\frac{1}{2}}} + \frac{e_i^1 - e_{i-1}^1}{h_{i-\frac{1}{2}}} + \alpha(e_i^1 - e_{i-1}^1) + \beta h_i e_i^1 = \gamma_i^1 - \gamma_{i-1}^1 + \delta_i^1, \tag{18}
$$

where

$$
\gamma_i^1 = -\frac{h_{i+1}^- - h_i^+}{2}\left(\alpha(\partial_1 u_i - u_x(x_{i+\frac{1}{2}})) + \beta(u_i - u(x_{i+\frac{1}{2}}))\right) - \alpha h_i^+(\partial_1 u_i - u_x(x_i)) + R^1_{i+\frac{1}{2}} + \alpha S_i^1. \tag{19}
$$

$$\delta_i^1 = -\frac{\beta}{2}\left((h_i^+)^2(\partial_1 u_i - u_x(x_i)) - (h_i^-)^2(\partial_1 u_{i-1} - u_x(x_{i-1}))\right) + \beta T_i^1. \tag{20}$$

Multiplying both sides of (18) by e_i^1 and summing from $i=1$ to $i=N$, we get

$$\begin{aligned} -\sum_{i=1}^N \frac{e_{i+1}^1 - e_i^1}{h_{i+\frac12}} e_i^1 \quad &+ \quad \sum_{i=1}^N \frac{e_i^1 - e_{i-1}^1}{h_{i-\frac12}} e_i^1 + \alpha \sum_{i=1}^N (e_i^1 - e_{i-1}^1) e_i^1 + \beta \sum_{i=1}^N h_i {e_i^1}^2 \\ &= \quad \sum_{i=1}^N \gamma_i^1 e_i^1 - \sum_{i=1}^N \gamma_{i-1}^1 e_i^1 + \sum_{i=1}^N \delta_i^1 e_i^1. \end{aligned} \tag{21}$$

This gives that

$$\sum_{0\le i\le N} \frac{(e_{i+1}^1 - e_i^1)^2}{h_{i+\frac12}} \le \left(\sum_{0\le i\le N} h_{i+\frac12} {\gamma_i}^2\right)^{\frac12} \left(\sum_{0\le i\le N} \frac{(e_{i+1}^1 - e_i^1)^2}{h_{i+\frac12}}\right)^{\frac12} + |\sum_{i=1}^N \delta_i^1 e_i^1|. \tag{22}$$

To estimate second term in the r.h.s of (22), we handle each term in (20). Indeed, we have by using inequality (14) and reordering sum of second term of (20), we can get

$$\begin{aligned} |\sum_{i=1}^N \delta_i^1 e_i^1| \quad &\le \quad c\left(\sum_{i=1}^N {h_i^+}^2 |\partial_1 u_i - u_x(x_i) e_i^1| + \sum_{i=1}^N {h_i^-}^2 |\partial_1 u_{i-1} - u_x(x_{i-1}) e_i^1| + \sum_{i=1}^N |T_i^1 e_i^1|\right) \\ &\le \quad c\left(h\sum_{i=1}^N h_i^+ |\partial_1 u_i - u_x(x_i) e_i^1| + h\sum_{i=1}^N h_i^- |\partial_1 u_{i-1} - u_x(x_{i-1}) e_i^1| + [\sum_{i=1}^N \frac{{T_i^1}^2}{h_i}]^{\frac12} [\sum_{i=1}^N h_i {e_i^1}^2]^{\frac12}\right) \\ &\le \quad c\{h[\sum_{i=1}^N h_i^+ (\partial_1 u_i - u_x(x_i))^2]^{\frac12} [\sum_{i=1}^N h_i^+ {e_i^1}^2]^{\frac12} + h[\sum_{i=1}^N h_i^- (\partial_1 u_{i-1} - u_x(x_{i-1}))^2]^{\frac12} [\sum_{i=1}^N h_i^- {e_i^1}^2]^{\frac12} \\ &\quad\quad + [\sum_{i=1}^N \frac{{T_i^1}^2}{h_i}]^{\frac12} [\sum_{i=1}^N h_i {e_i^1}^2]^{\frac12}\}. \end{aligned}$$

But

$$\left(\sum_{i=1}^N h_i^+ (\partial_1 u_i - u_x(x_i))^2\right)^{\frac12} \le \left(\sum_{0\le i\le N} h_{i+\frac12} (\partial_1 u_i - u_x(x_i))^2\right)^{\frac12}, \tag{23}$$

and

$$\begin{aligned} \left(\sum_{i=1}^N h_i^- (\partial_1 u_{i-1} - u_x(x_{i-1}))^2\right)^{\frac12} \quad &= \quad \left(\sum_{i=0}^{N-1} h_{i+1}^- (\partial_1 u_i - u_x(x_i))^2\right)^{\frac12} \\ &\le \quad \left(\sum_{0\le i\le N} h_{i+\frac12} (\partial_1 u_i - u_x(x_i))^2\right)^{\frac12}. \end{aligned} \tag{24}$$

Therefore

$$|\sum_{i=1}^N \delta_i^1 e_i^1| \le c\left(h\left(\sum_{0\le i\le N} h_{i+\frac12} (\partial_1 u_i - u_x(x_i))^2\right)^{\frac12} + \left(\sum_{i=1}^N \frac{{T_i^1}^2}{h_i}\right)^{\frac12}\right)\left(\sum_{i=1}^N h_i {e_i^1}^2\right)^{\frac12}. \tag{25}$$

Inequality (25) combined with discrete Poincaré and triangular inequalities imply the following inequality

$$
\begin{aligned}
\left(\sum_{0\le i\le N}\frac{(e^1_{i+1}-e^1_i)^2}{h_{i+\frac12}}\right)^{\frac12} \quad &\le \quad \left(\sum_{0\le i\le N} h_{i+\frac12}\left(\frac{h^-_{i+1}-h^+_i}{2}\right)^2\alpha^2(\partial_1 u_i - u_x(x_{i+\frac12}))^2\right)^{\frac12}\\
&+ \quad \left(\sum_{0\le i\le N} h_{i+\frac12}\left(\frac{h^-_{i+1}-h^+_i}{2}\right)^2\beta^2(u_i - u(x_{i+\frac12}))^2\right)^{\frac12}\\
&+ \quad \left(\sum_{0\le i\le N} \alpha^2 h_{i+\frac12}{h^+_i}^2(\partial_1 u_i - u_x(x_{i+\frac12}))^2\right)^{\frac12}\\
&+ \quad \left(\sum_{0\le i\le N} h_{i+\frac12}(R^1_{i+\frac12})^2\right)^{\frac12} + \left(\sum_{0\le i\le N} \alpha^2 h_{i+\frac12}(S^1_i)^2\right)^{\frac12}\\
&+ \quad c\left(h\left(\sum_{0\le i\le N} h_{i+\frac12}(\partial_1 u_i - u_x(x_i))^2\right)^{\frac12} + \left(\sum_{i=1}^{N}\frac{{T^1_i}^2}{h_i}\right)^{\frac12}\right)
\end{aligned}
\tag{26}
$$

To estimate the r.h.s of inequality (26), we need the following estimates

Lemma 2.1. *Let $u^h = (u_i)$ be the basic solution defined by (2), the following estimates hold*

1. $\left(\sum_{0\le i\le N} h_{i+\frac12}(u_i - u(x_{i+\frac12}))^2\right)^{\frac12} \le ch\|u_{2x}\|_{\infty,\bar I}$.

2. $\left(\sum_{0\le i\le N} h_{i+\frac12}(\partial_1 u_i - u_x(x_{i+\frac12}))^2\right)^{\frac12} \le ch\|u_{2x}\|_{\infty,\bar I}$.

3. $\left(\sum_{0\le i\le N} h_{i+\frac12}(\partial_1 u_i - u_x(x_i))^2\right)^{\frac12} \le ch\|u_{2x}\|_{\infty,\bar I}$

4. $\left(\sum_{0\le i\le N} h_{i+\frac12}{R^1_{i+\frac12}}^2\right)^{\frac12} \le ch^2\|u_{3x}\|_{\infty,\bar I}$.

5. $\left(\sum_{0\le i\le N} h_{i+\frac12}{S^1_i}^2\right)^{\frac12} \le ch^2\|u_{2x}\|_{\infty,\bar I}$.

6. $\left(\sum_{i=1}^{N}\frac{{T^1_i}^2}{h_i}\right)^{\frac12} \le ch^2\|u_{2x}\|_{\infty,\bar I}$.

Remark 2.3. *Estimate* **1.** *of lemma 2.1 can be done through uniform estimate (6), where the estimate (6) holds for one dimension (see remark 2.7, page 18 in* [7]*). The proof, we wish to present, holds also for the case of the finite volume scheme of five points in two dimensional space.*

Proof.

1. By triangular inequality , we have

$$
\begin{aligned}
\Big(\sum_{0\le i\le N} h_{i+\frac12}(u_i-u(x_{i+\frac12}))^2\Big)^{\frac12} &\le \Big(\sum_{0\le i\le N} h_{i+\frac12}(u_i-u(x_i))^2\Big)^{\frac12}+\Big(\sum_{0\le i\le N} h_{i+\frac12}(u(x_i)-u(x_{i+\frac12}))^2\Big)^{\frac12}\\
&\le \left(\sum_{0\le i\le N} h_{i+\frac12}e_i^2\right)^{\frac12}+ch\|u_{2x}\|_{\infty,\bar I}, \qquad (27)
\end{aligned}
$$

using the fact that $e_0=0$ and $\sum_{0\le i\le N} h_{i+\frac12}=1$ combined with Cauchy-Schwarz inequality to get

$$
\begin{aligned}
\left(\sum_{0\le i\le N} h_{i+\frac12}e_i^2\right)^{\frac12} &\le \left(\sum_{0\le i\le N}\frac{(e_{i+1}-e_i)^2}{h_{i+\frac12}}\right)^{\frac12}\\
&\le ch\|u_{2x}\|_{\infty,\bar I}.
\end{aligned}
$$

this with (27) imply the desired inequality 1 of lemma.

2. Using the same thecnique, yields

$$
\begin{aligned}
\left(\sum_{0\le i\le N} h_{i+\frac12}(\partial_1 u_i-u_x(x_{i+\frac12}))^2\right)^{\frac12} &\le \left(\sum_{0\le i\le N} h_{i+\frac12}(\partial_1 u_i-\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac12}})^2\right)^{\frac12}\\
&+\left(\sum_{0\le i\le N} h_{i+\frac12}(\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac12}}-u_x(x_{i+\frac12}))^2\right)^{\frac12}\\
&\le ch\|u_{2x}\|_{\infty,\bar I}.
\end{aligned}
$$

3. can be obtained as done for **1.** and **2.**

4. according to inequalty (10), we have $|R^1_{i+\frac12}|\le ch^2$, this implies the inequality 4 of the lemma.

5. according to (12), S^1_i is of order $O(h^2)$ in uniform norm, which implies 5 of the lemma.

6. according to (14), $\frac{{T^1_i}^2}{h_i}\le ch^4h_i$, the inequality 6 of lemma will be obvious. □

Coming back now to the lemma 2.1, since $h^-_{i+1}-h^+_i=O(h)$, then we have the following $O(h)$ improvement.

Theorem 2.2. *If the unknown solution u of 1 belonging to $C^3(\bar I)$. Then the error in the first correction defined by 17 is of order $O(h^2)$ in the discrete H^1_0 norm, i.e*

$$
\left(\sum_{0\le i\le N}\frac{(e^1_{i+1}-e^1_i)^2}{h_{i+\frac12}}\right)^{\frac12}\le ch^2\|u\|_{3,\infty,\bar I}, \qquad (28)
$$

where $e^1_i=u^1_i-u(x_i)$ and $(u^1_i)_i$ are the components of the first correction defined by (17).

2.3.1. **Other Variant to Estimate the Second Derivative of Unknown Solution**. For some cases, like the model $-u_{2x}+\beta u=f$, we have other possibility to approximate the second derivative u_{2x} of u. Indeed, u_{2x} satisfies the following equation

$$
\begin{cases}
-v_{xx}+\beta v=f_{2x}(x), x\in I=(0,1),\\
v(0)=-f(0),\\
v(1)=-f(1).
\end{cases}
$$

Then u_{2x} satisfies the same equation that is satisfying by u, this allowing us to get a finite volume approximation to u_{2x}, provided that $u \in C^4(\bar{I})$ (see theorem 2.1), by using the same scheme that used to compute the basic solution u^h, more precisely, we use the same matrix, that used to compute u^h, to compute a finite volume approximation to u_{2x} . This idea can be used also to compute higher order of corrections.

2.4. **Second Correction.** The situation in the second correction is different to that of the first correction, because it is easy to pass from the derivative into its forward approximation by an order of convergence $O(h)$ (see lemma 2.1). To get the second correction of order $O(h^3)$, we have to look for approximations of first and second derivative of the unknown solution, of orders $O(h^2)$. That is why, we discribe how to overcome this difficuly.

Assuming that $u \in C^4(\bar{I})$, by similar way to that one used to compute an expansion for the error (21), we can get

$$
\begin{aligned}
-\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac12}} &+ \frac{u(x_i)-u(x_{i-1})}{h_{i-\frac12}} + \alpha u(x_i) - \alpha u(x_{i-1}) + \beta h_i u(x_i) = \int_{K_i} f dx \\
&- \frac{h_{i+1}^- - h_i^+}{2} u_{2x}(x_{i+\frac12}) - \frac{{h_{i+1}^-}^2 - h_{i+1}^- h_i^+ + {h_i^+}^2}{6} u_{3x}(x_{i+\frac12}) \\
&+ \frac{h_i^- - h_{i-1}^+}{2} u_{2x}(x_{i-\frac12}) + \frac{{h_i^-}^2 - h_i^- h_{i-1}^+ + {h_{i-1}^+}^2}{6} u_{3x}(x_{i-\frac12}) \\
&- \alpha h_i^+ u_x(x_i) - \alpha \frac{{h_i^+}^2}{2} u_{2x}(x_i) + \alpha h_{i-1}^+ u_x(x_{i-1}) + \alpha \frac{{h_{i-1}^+}^2}{2} u_{2x}(x_{i-1}) \\
&- \frac{\beta}{2} \left({h_i^+}^2 u_x(x_i) - {h_i^-}^2 u_x(x_{i-1}) \right) - \frac{\beta}{6} \left({h_i^+}^3 u_{2x}(x_i) + {h_i^-}^3 u_{2x}(x_{i-1}) \right) \\
&+ \frac{\beta}{2} {h_i^-}^2 h_{i-\frac12} u_{2x}(x_{i-1}) - R^2_{i+\frac12} + R^2_{i-\frac12} - \alpha S_i^2 + \alpha S_{i-1}^2 - \beta T_i^2,
\end{aligned} \tag{29}
$$

where

$$ |R^2_{i+\frac12}| \le c h^3_{i+\frac12} \|u\|_{4,\infty,\bar{I}}, \ \forall i \in \{0, ..., N\}, \tag{30} $$

$$ |S_i^2| \le c {h_i^+}^3 \|u\|_{3,\infty,\bar{I}}, \ \forall i \in \{0, ..., N\}, \tag{31} $$

$$ |T_i^2| \le c h^3 h_i \|u\|_{3,\infty,\bar{I}}, \ \forall i \in \{1, ..., N\}, \tag{32} $$

In order to get correction of order $O(h^3)$ taking into account the coefficients of the pointwise derivatives in the r.h.s of (29), we have to find approximations of order $O(h^2)$ to pointwise first and second derivative, and $O(h)$ to the pointwise third derivative in discrete L^2-norm.
Begining by the pointwise second derivative $u_{2x}(x_i)$, and looking for approxomation $u^{h,2}_{2x} = (u^{h,2}_{2x})_i, i \in \{0, ..., N\}$, the idea that we want to suggest, is based on the use of Taylor's formula and values of the first correction. Indeed

$$
\begin{aligned}
u_{2x}(x_i) &= \alpha u(x_i) + \beta u_x(x_i) + f(x_i) \\
&= \alpha u(x_i) + \beta \left(\frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac12}} - \frac{h_{i+\frac12}}{2} u_{2x}(x_i) + r_{i+\frac12} \right) + f(x_i),
\end{aligned} \tag{33}
$$

where

$$ r_{i+\frac12} \le c h^2_{i+\frac12} |u_{3x}|_\infty . \tag{34} $$

Let δ_i^h be the positive number $1+\frac{\beta}{2}h_{i+\frac{1}{2}}$, the equation (33) becomes as

$$\delta_i^h u_{2x}(x_i)=\alpha u(x_i)+\beta\left(\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}}\right)+f(x_i)+\beta r_{i+\frac{1}{2}}. \tag{35}$$

Because of the trivial inequality $1\le\delta_i^h\le 1+\frac{\beta}{2}$, we can suggest the following approximation $u_{2x}(x_i)$

$$(u_{2x}^{h,2})_i=\frac{\alpha}{\delta_i^h}u_i^1+\frac{\beta}{\delta_i^h}\left(\frac{u_{i+1}^1-u_i^1}{h_{i+\frac{1}{2}}}\right)+\frac{f(x_i)}{\delta_i^h},\ \forall i\in\{0,...,N\}. \tag{36}$$

Looking, now, for an approximation $u_{3x}^{h,2}=(u_{3x}^{h,2})_i,\{0,...,N\}$ to pointwise third derivative. Because of $u_{3x}=\alpha\beta u+(\alpha^2+\beta)u_x-f_x$, it is useful to suggest the following approximation

$$(u_{3x}^{h,2})_i=\alpha\beta u_i^1+(\alpha^2+\beta)\left(\frac{u_{i+1}^1-u_i^1}{h_{i+\frac{1}{2}}}\right)-f_x(x_i),\forall i\in\{0,...,N\}. \tag{37}$$

Remark 2.4. *We can use the approximation of the second derivative, that used in the first correction, to compute its approximation to obtain second correction.*

We shall prove now the following lemma

Lemma 2.2. *If the solution u of the equation (1) belonging to $C^3(\bar{I})$ and $u_{2x}^{h,2}$,$u_{3x}^{h,2}$ be the discrete expansions defined respectively by (36) and (37). Then the following estimates hold*

1. $\left(\sum_{i=0}^{N}h_{i+\frac{1}{2}}((u_{2x}^{h,2})_i-u_{2x}(x_i))^2\right)^{\frac{1}{2}}\le ch^2\|u\|_{3,\infty,\bar{I}}.$

2. $\left(\sum_{i=0}^{N}h_{i+\frac{1}{2}}((u_{3x}^{h,2})_i-u_{3x}(x_i))^2\right)^{\frac{1}{2}}\le ch\|u\|_{3,\infty,\bar{I}}..$

Proof. Substracting (36) from (35), to get

$$u_{2x}(x_i)-(u_{2x}^{h,2})_i = \frac{\alpha}{\delta_i^h}(u(x_i)-u_i^1)+\frac{\beta}{\delta_i^h}\left(\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}}-\frac{u_{i+1}^1-u_i^1}{h_{i+\frac{1}{2}}}\right)+\frac{\beta r_{i+\frac{1}{2}}}{\delta_i^h}.$$

This implies , using triangular inequality with bound uniform of δ_i^h, that

$$\begin{aligned}\left(\sum_{i=0}^{N}h_{i+\frac{1}{2}}((u_{2x}^{h,2})_i-u_{2x}(x_i))^2\right)^{\frac{1}{2}} \le\ & c[\left(\sum_{i=0}^{N}h_{i+\frac{1}{2}}(u(x_i)-u_i^1)^2\right)^{\frac{1}{2}}\\ &+\left(\sum_{i=0}^{N}h_{i+\frac{1}{2}}(\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}}-\frac{u_{i+1}^1-u_i^1}{h_{i+\frac{1}{2}}})^2\right)^{\frac{1}{2}}\\ &+\left(\sum_{i=0}^{N}h_{i+\frac{1}{2}}r_{i+\frac{1}{2}}^2\right)^{\frac{1}{2}}]\end{aligned} \tag{38}$$

Using inequalities (28) and (34) to get the desired estimation 1 of lemma 2.2.
By the same way, we can prove the second inequality. □
After having acheived optimal approximations for the pointwise second and third derivative, we look now for optimal approximations for $(u_{2x}(x_{i+\frac{1}{2}}))_i$, $(u_{3x}(x_{i+\frac{1}{2}}))_i$, $(u_x(x_i)_i$, $\forall i\in\{0,...,N\}$, we

have $u_{2x}(x_{i+\frac{1}{2}}) = u_{2x}(x_i) + h_i^+ u_{3x}(x_i) + s_i$, where $|s_i| \le c{h_i^+}^2 \|u_{4x}\|_\infty$, this allows us to suggest the folowing approximation for $(u_{2x}(x_{i+\frac{1}{2}}))_i$

$$u_{2x}^{i+\frac{1}{2},2} = (u_{2x}^{h,2})_i + h_i^+ (u_{3x}^{h,2})_i, \tag{39}$$

for pointwise third derivative, we can suggest the following approximation

$$u_{3x}^{i+\frac{1}{2},2} = (u_{3x}^{h,2})_i, \tag{40}$$

for pointwise first derivative, we can use the trick that used for pointwise second derivative

$$u(x_i) = \frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} - \frac{h_{i+\frac{1}{2}}}{2} u_{2x}(x_i) + t_i, \tag{41}$$

where $|t_i| \le c h_{i+\frac{1}{2}}^2 \|u_{3x}|_\infty$, and an approximation will be suggested as follows

$$(u_x^{h,2})_i = \frac{u_{i+1}^1 - u_i^1}{h_{i+\frac{1}{2}}} - \frac{h_{i+\frac{1}{2}}}{2} (u_{2x}^{h,2})_i, \tag{42}$$

We would now prove the following lemma

Lemma 2.3. *If the solution u of equation 1 belonging to $C^4(\bar{I})$. Then the approximations $(u_{2x}^{i+\frac{1}{2},2})_{i=0}^N$, $(u_{3x}^{i+\frac{1}{2},2})_{i=0}^N$ and $((u_x^{h,2})_i)_{i=0}^N$ defined respectively by the expansions (39), (40) and (42) satisfying the following estimates*

1. $\displaystyle\left(\sum_{i=0}^N h_{i+\frac{1}{2}} (u_{2x}^{i+\frac{1}{2},2} - u_{2x}(x_{i+\frac{1}{2}}))^2\right)^{\frac{1}{2}} \le ch^2 \|u\|_{4,\infty,\bar{I}}.$
2. $\displaystyle\left(\sum_{i=0}^N h_{i+\frac{1}{2}} (u_{3x}^{i+\frac{1}{2},2} - u_{3x}(x_{i+\frac{1}{2}}))^2\right)^{\frac{1}{2}} \le ch \|u\|_{4,\infty,\bar{I}}.$
3. $\displaystyle\left(\sum_{i=0}^N h_{i+\frac{1}{2}} ((u_x^{h,2})_i - u_x(x_i))^2\right)^{\frac{1}{2}} \le ch^2 \|u\|_{3,\infty,\bar{I}}.$

Proof.

1. We proceed as done in the proof of lemma 2.2. Using triangular inequality and equality (39), to obtain

$$\begin{aligned}\left(\sum_{i=0}^N h_{i+\frac{1}{2}} (u_{2x}^{i+\frac{1}{2},2} - u_{2x}(x_{i+\frac{1}{2}}))^2\right)^{\frac{1}{2}} \quad &\le \quad \left(\sum_{i=0}^N h_{i+\frac{1}{2}} ((u_{2x}^{h,2})_i - u_{2x}(x_i))^2\right)^{\frac{1}{2}} \\ &+ \left(\sum_{i=0}^N h_{i+\frac{1}{2}} {h_i^+}^2 ((u_{3x}^{h,2})_i - u_{3x}(x_i))^2\right)^{\frac{1}{2}} \\ &+ \left(\sum_{i=0}^N h_{i+\frac{1}{2}} s_i^2\right)^{\frac{1}{2}}\end{aligned}$$

Using lemma 2.2 and the uniform bound of s_i to obtain the desired inequality 1 of the lemma 2.3.

2. and **3.** of the lemma can be handled by the same way as done for the first estimation. □

Now we are able to define the second correction $u_2^h = (u_i^2)_{i=0}^{N+1}$, where $u_0^2 = u_{N+1}^2 = 0$ and for all $i \in \{1, ..., N\}$, we have

$$
\begin{aligned}
-\frac{u_{i+1}^2 - u_i^2}{h_{i+\frac{1}{2}}} &+ \frac{u_i^2 - u_{i-1}^2}{h_{i-\frac{1}{2}}} + \alpha u_i^2 - \alpha u_{i-1}^2 + \beta h_i u_i^2 = \int_{K_i} f dx \\
&- \frac{h_{i+1}^- - h_i^+}{2} u_{2x}^{i+\frac{1}{2},2} - \frac{{h_{i+1}^-}^2 - h_{i+1}^- h_i^+ + {h_i^+}^2}{6} u_{3x}^{i+\frac{1}{2},2} \\
&+ \frac{h_i^- - h_{i-1}^+}{2} u_{2x}^{i-\frac{1}{2},2} + \frac{{h_i^-}^2 - h_i^- h_{i-1}^+ + {h_{i-1}^+}^2}{6} u_{3x}^{i-\frac{1}{2},2} \\
&- \alpha h_i^+ (u_x^{h,2})_i - \alpha \frac{{h_i^+}^2}{2} (u_{2x}^{h,2})_i + \alpha h_{i-1}^+ (u_x^{h,2})_{i-1} + \alpha \frac{{h_{i-1}^+}^2}{2} (u_{2x}^{h,2})_{i-1} \\
&- \frac{\beta}{2} \{ {h_i^+}^2 (u_x^{h,2})_i - {h_i^-}^2 (u_x^{h,2})_{i-1} \} - \frac{\beta}{6} \{ {h_i^+}^3 (u_{2x}^{h,2})_i + {h_i^-}^3 (u_{2x}^{h,2})_{i-1} \} \\
&+ \frac{\beta}{2} {h_i^-}^2 h_{i-\frac{1}{2}} (u_{2x}^{h,2})_{i-1}
\end{aligned} \tag{43}
$$

To analyse the error of the convergence, we proceed as done for the basic solution and the first correction. Indeed, let $e_i^2 = u_i^2 - u(x_i)$ and substracting equality (29) from (43) to get

$$
-\frac{e_{i+1}^2 - e_i^2}{h_{i+\frac{1}{2}}} + \frac{e_i^2 - e_{i-1}^2}{h_{i-\frac{1}{2}}} + \alpha e_i^2 - \alpha e_{i-1}^2 + \beta h_i e_i^2 = \gamma_i^2 - \gamma_{i-1}^2 + \delta_i^2. \tag{44}
$$

where

$$
\begin{aligned}
\gamma_i^2 &= -\frac{h_{i+1}^- - h_i^+}{2} (u_{2x}^{i+\frac{1}{2},2} - u_{2x}(x_{i+\frac{1}{2}})) - \frac{{h_{i+1}^-}^2 - h_{i+1}^- h_i^+ + {h_i^+}^2}{6} (u_{3x}^{i+\frac{1}{2},2} - u_{3x}(x_{i+\frac{1}{2}})) \\
&- \alpha h_i^+ ((u_x^{h,2})_i - u_x(x_i)) - \alpha \frac{{h_i^+}^2}{2} ((u_{2x}^{h,2})_i - u_{2x}(x_i)) + R_{i+\frac{1}{2}}^2 + \alpha S_i^2
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\delta_i^2 &= -\frac{\beta}{2} \{ {h_i^+}^2 ((u_x^{h,2})_i - u_x(x_i)) - {h_i^-}^2 ((u_x^{h,2})_{i-1} - u_x(x_{i-1})) \} \\
&- \frac{\beta}{6} \{ {h_i^+}^3 (u_{2x}^{h,2})_i - u_{2x}(x_i)) + {h_i^-}^3 (u_{2x}^{h,2})_{i-1} - u_{2x}(x_{i-1})) \} \\
&- \frac{\beta}{2} {h_i^-}^2 h_{i-\frac{1}{2}} ((u_{2x}^{h,2})_{i-1} - u_{2x}(x_{i-1})) + \beta T_i^2
\end{aligned} \tag{46}
$$

Using the proof of the convergence of the basic solution and the first correction together with triangular and discrete Poincare inequalities combined with lemma 2.3 and expansions (30), (31) and (32) to get the following $O(h^2)$ improvement.

Theorem 2.3. *If the unknown solution of (1) belonging to $C^4(\bar{I})$. Then the error in the second correction defined by (43) is of order $O(h^3)$ in the discrete H_0^1 norm, i.e.*

$$
\left(\sum_{0 \le i \le N} \frac{(e_{i+1}^2 - e_i^2)^2}{h_{i+\frac{1}{2}}} \right)^{\frac{1}{2}} \le c h^3 \|u\|_{4,\infty,\bar{I}} \tag{47}
$$

where $e_i^2 = u_i^2 - u(x_i)$ and $(u_i^2)_i$ are the components of the second correction defined by (43).

2.5. **Corrections of Higher Order.** In this section, we give the general formulation of an arbitrary correction. The pointwise derivatives will be approximated in the light of ones of the first and second correction. The proof of the order of the convergence is the same one that done for the first and second correction.
For each integer $k \geq 1$

$$\begin{aligned} -\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}}+\frac{u(x_i)-u(x_{i-1})}{h_{i-\frac{1}{2}}} &= -u_x(x_{i+\frac{1}{2}})+u_x(x_{i-\frac{1}{2}})+\sum_{m=2}^{k+1} a_m^i u_{mx}(x_{i+\frac{1}{2}}) \\ &- \sum_{m=2}^{k+1} a_m^{i-1} u_{mx}(x_{i-\frac{1}{2}}) - R^k_{i+\frac{1}{2}} + R^k_{i-\frac{1}{2}} \end{aligned}$$

$$a^i_m = \frac{1}{m!}\left(\sum_{j=0}^{m-1} (h^-_{i+1})^j (-h^+_i)^{m-1-j}\right). \tag{48}$$

and

$$|R_{i+\frac{1}{2}}| \leq c h^{k+1}_{i+\frac{1}{2}} \|u\|_{k+2,\infty,\bar{I}} \tag{49}$$

We have also

$$u(x_{i+\frac{1}{2}}) = u(x_i) + \sum_{m=1}^{k} \frac{(h^+_i)^m}{m!} u_{mx}(x_i) + S^k_i, \tag{50}$$

where

$$|S^k_i| \leq c h^{k+1}_i \|u\|_{k+1,\infty,\bar{I}} \tag{51}$$

For the fifth term in (8), we have

$$\int_{K_i} u(x)dx = h_i u(x_i) + \sum_{m=1}^{k} \frac{(h^+_i)^{m+1}}{(m+1)!} u_{mx}(x_i) - \sum_{m=1}^{k} \frac{(-h^-_i)^{m+1}}{(m+1)!} \sum_{j=0}^{k-m} \frac{h^j_{i-\frac{1}{2}}}{j!} u_{(m+j)x}(x_{i-1}) + T^k_i, \tag{52}$$

where

$$|T^k_i| \leq c h_i h^{k+1} \|u\|_{k+1,\infty,\bar{I}} \tag{53}$$

Substituting terms of (8) by theirs expansions (50), (51) and (53), we obtain

$$\begin{aligned} -\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}}+\frac{u(x_i)-u(x_{i-1})}{h_{i-\frac{1}{2}}} &+ \alpha u(x_i) - \alpha u(x_{i-1}) + \beta h_i u(x_i) = \int_{K_i} f dx \\ &- \sum_{m=2}^{k+1} a^i_m u_{mx}(x_{i+\frac{1}{2}}) + \sum_{m=2}^{k+1} a^{i-1}_m u_{mx}(x_{i-\frac{1}{2}}) \\ &- \alpha \sum_{m=1}^{k} \frac{(h^+_i)^m}{m!} u_{mx}(x_i) + \alpha \sum_{m=1}^{k} \frac{(h^+_{i-1})^m}{m!} u_{mx}(x_{i-1}) \\ &- \beta\{\sum_{m=1}^{k} \frac{(h^+_i)^{m+1}}{(m+1)!} u_{mx}(x_i) - \sum_{m=1}^{k} \frac{(-h^-_i)^{m+1}}{(m+1)!} \sum_{j=0}^{k-m} \frac{h^j_{i-\frac{1}{2}}}{j!} u_{(m+j)x}(x_{i-1})\} \\ &- R^k_{i+\frac{1}{2}} + R^k_{i-\frac{1}{2}} - \alpha S^k_i + \alpha S^k_{i-1} - \beta T^k_i \end{aligned} \tag{54}$$

After having found an expansion approximating the equation (8), we need now the following useful lemma

Lemma 2.4. *Each kth derivative of the solution u of the problem 1 can be expanded as a linear combination of the solution itself, its derivative and the derivatives of the given function f up to and including $(k-2)th$ derivative, i.e, there exist reals $\{\alpha_j^k\}_{j=0}^{k-2} \cup \{\overline{\alpha}_1^k, \overline{\alpha}_2^k\}$ such that*

(55) $$u_{kx} = \sum_{j=0}^{k-2} \alpha_j^k f_{jx} + \overline{\alpha}_1^k u + \overline{\alpha}_2^k u_x.$$

Proof. We can prove this lemma by induction on the integer k. $\square$
Assuming, now, that we have obtained the $(k-1)th$ correction $u_{k-1}^h = (u_i^{k-1})_i$, i.e. approximation of order $O(h^k)$. According to equality (54), to obtain correction of order $O(h^{k+1})$, we have to find approximations for the pointwise derivative up and including $k+1$ order of the solution u. The idea which we will present is similar to that one presented to compute second correction.

At first, we look for optimal approximations to $u_{2x}(x_i)$,..., $u_{(k+1)x}(x_i)$. To do so, we use the previous correction, i.e. $(k-1)th$ correction, and the optimal approximations to $u_{2x}(x_i)$,...,$u_{kx}(x_i)$ used to define this correction. That is why, we define the kth correction by induction, we assume that, we have obtained $(k-1)th$ correction of order $O(h^k)$ and we have found optimal approximations (according to theirs coefficients in (54) $(u_{2x}^{h,k-1})_i$,...,$(u_{kx}^{h,k-1})_i$ for $u_{2x}(x_i)$,..., $u_{kx}(x_i)$, i.e. theirs orders of convergence are $O(h^{k-1})$,...,$O(h)$.

Because the coefficient of $u_{(k+1)x}(x_i)$ in (54) is of order $O(h^k)$, it suffices to approximate it by in order $O(h)$. This, can be done easily through lemma 2.4,i.e. an approximation defined by

(56) $$(u_{(k+1)x}^{h,k})_i = \sum_{j=0}^{k-1} \alpha_j^{k+1} f_{jx}(x_i) + \bar{\alpha}_1^{k+1} u_i^{k-1} + \bar{\alpha}_2^{k+1} \partial u_i^{k-1}, \forall i \in \{0, ..., N\}.$$

We can use, also, in (56) instead of the the $(k-1)$ correction, the basic solution u^h.
For any integer β such that $2 \le \beta \le k$, we look to find approximation $u_{\beta x}^{h,k}$ of order $O(h^{k+2-\beta})$ to pointwise derivative $(u_{\beta x}(x_i))_i$ of order β, bacause the coefficients of such derivative in (54) are of order $\beta - 1$. We have through lemma 2.4

$$\begin{aligned} u_{\beta x}(x_i) &= \sum_{j=0}^{\beta-2} \alpha_j^\beta f_{jx}(x_i) + \bar{\alpha}_1^\beta u(x_i) + \bar{\alpha}_2^\beta u_x(u_i) \\ &= \sum_{j=0}^{\beta-1} \alpha_j^\beta f_{jx}(x_i) + \bar{\alpha}_1^\beta u(x_i) \\ &+ \bar{\alpha}_2^\beta \left(\frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} - \sum_{j=2}^{k-\beta+2} \frac{u_{jx}(x_i)}{j!} h_{i+\frac{1}{2}}^{j-1} \right) + r_i^k, \ \forall i \in \{0, ..., N\}, \end{aligned} \tag{57}$$

where

(58) $$|r_i^k| \le c h_{i+\frac{1}{2}}^{k+2-\beta} |u|_{k+3-\beta,\infty}$$

An obvious approximation for pointwise derivative of order β can be given as

(59) $$(u_{\beta x}^{h,k})_i = \sum_{j=0}^{\beta-2} \alpha_j^\beta f_{jx}(x_i) + \bar{\alpha}_1^\beta u_i^{k-1} + \bar{\alpha}_2^\beta \left(\frac{u_{i+1}^{k-1} - u_i^{k-1}}{h_{i+\frac{1}{2}}} - \sum_{j=2}^{k-\beta+2} \frac{(u_{jx}^{h,k-1})_i}{j!} h_{i+\frac{1}{2}}^{j-1} \right),$$

We would prove the following lemma

Lemma 2.5. *If the solution u of the equation 1 belonging to $C^{k+1}(\bar{I})$. Then the approximations $u^{h,k}_{\beta x}$, where $2 \leq \beta \leq k+1$, defined by the expansions (56) and (59) satisfying the following estimate*

$$\left(\sum_{i=0}^{N} h_{i+\frac{1}{2}}((u^{h,k}_{\beta x})_i - u_{\beta x}(x_i))^2\right)^{\frac{1}{2}} \leq ch^{k-\beta+2}\|u\|_{k+1,\infty,\bar{I}}. \tag{60}$$

Proof . We can prove this by induction. □

After having found optimal approximations to fundamental pointwise derivatives, we derive now optimal approximations for $(u_{\beta x}(x_{i+\frac{1}{2}}))_{i=0}^{N}$,$2 \leq \beta \leq k+1$ and $(u_x(x_i))_{i=0}^{N}$.

We have

$$u_{\beta x}(x_{i+\frac{1}{2}}) = \sum_{j=0}^{k-\beta+1} \frac{(h_i^+)^j}{j!} u_{(\beta+j)x}(x_i) + s_i^k, \forall i \in \{0, ..., N\} \tag{61}$$

where

$$|s_i^k| \leq c(h_i^+)^{k-\beta+2}|u_{(k+2)x}|_\infty \tag{62}$$

We can suggest the following approximation

$$(u^{i+\frac{1}{2},k}_{\beta x})_i = \sum_{j=0}^{k-\beta+1} \frac{(h_+)^j}{j!}(u^{h,k}_{(\beta+j)x})_i, \forall i \in \{0, ..., N\}. \tag{63}$$

For the pointwise first derivative, we can do

$$u_x(x_i) = \frac{u(x_{i+1}) - u(x_i)}{h_{i+\frac{1}{2}}} - \sum_{j=2}^{k} \frac{h^{j-1}_{i+\frac{1}{2}}}{j!} u_{jx}(x_i) + t_i^k, \tag{64}$$

where

$$|t_i^k| \leq ch^k_{i+\frac{1}{2}}|u_{(k+1)x}|_\infty, \forall i \in \{0, ..., N\} \tag{65}$$

this allowing us to consider the following approximation

$$(u^{h,k}_x)_i = \frac{u^{k-1}_{i+1} - u^{k-1}_i}{h_{i+\frac{1}{2}}} - \sum_{j=2}^{k} \frac{h^{j-1}_{i+\frac{1}{2}}}{j!}(u^{h,k}_{jx})_i, \forall i \in \{0, ..., N\} \tag{66}$$

We need the following useful lemma

Lemma 2.6. *If the solution u of the equation (1) belonging to $C^{k+1}(\bar{I})$. Then the approximations $(u^{i+\frac{1}{2},k}_{\beta x})_i$, where $2 \leq \beta \leq k+1$, defined and $(u^{h,k}_x)_i$ defined respectively by the expansions (63) and (66) satisfying the following estimation*

$$\left(\sum_{i=0}^{N} h_{i+\frac{1}{2}}((u^{i+\frac{1}{2},k}_{\beta x})_i - u_{\beta x}(x_{i+\frac{1}{2}}))^2\right)^{\frac{1}{2}} \leq ch^{k-\beta+2}\|u\|_{k+1,\infty,\bar{I}}.$$

$$\left(\sum_{i=0}^{N} h_{i+\frac{1}{2}}((u^{h,k}_x)_i - u_x(x_i))^2\right)^{\frac{1}{2}} \leq ch^{k}\|u\|_{k+1,\infty,\bar{I}}.$$

Proof. The proof can be done as done for proving lemma 2.3. □
Now we are able to define the kth correction $u_h^k = (u_i^k)_{i=0}^{N+1}$, where $u_0^k = u_{N+1}^k = 0$ and for $i \in \{1, ..., N\}$, we have

$$
\begin{aligned}
-\frac{u_{i+1}^k - u_i^k}{h_{i+\frac{1}{2}}} + \frac{u_i^k - u_{i-1}^k}{h_{i-\frac{1}{2}}} + \alpha u_i^k - \alpha u_{i-1}^k + \beta h_i u_i^k &= \int_{K_i} f dx \\
&- \sum_{m=2}^{k+1} a_m^i (u_{mx}^{i+\frac{1}{2},k})_i + \sum_{m=2}^{k+1} a_m^{i-1} u_{mx}^{i-\frac{1}{2},k} - \alpha \sum_{m=1}^{k} \frac{(h_i^+)^m}{m!} (u_{mx}^{h,k})_i + \alpha \sum_{m=1}^{k} \frac{(h_{i-1}^+)^m}{m!} (u_{mx}^{h,k})_{i-1}) \\
&-\beta \left(\sum_{m=1}^{k} \frac{(h_i^+)^{m+1}}{(m+1)!} (u_{mx}^{h,k})_i - \sum_{m=1}^{k} \frac{(-h_i^-)^{m+1}}{(m+1)!} \sum_{j=0}^{k-m} \frac{h_{i-\frac{1}{2}}^j}{j!} (u_{(m+j)x})_{i-1}^{h,k} \right)
\end{aligned}
\tag{67}
$$

Consedering the following expansions

$$
\gamma_i^k = - \sum_{m=2}^{k+1} a_m^i ((u_{mx}^{i+\frac{1}{2},k})_i - u_{mx}(x_{i+\frac{1}{2}})) - \alpha \sum_{m=1}^{k} \frac{(h_i^+)^m}{m!} ((u_{mx}^{h,k})_i - u_{mx}(x_i)) + R_{i+\frac{1}{2}}^k + \alpha S_i^k \tag{68}
$$

(69)

$$
\delta_i^k = -\beta \left(\sum_{m=1}^{k} \frac{(h_i^+)^{m+1}}{(m+1)!} \left((u_{mx}^{h,k})_i - u_{mx}(x_i) \right) - \sum_{m=1}^{k} \frac{(-h_i^-)^{m+1}}{(m+1)!} \sum_{j=0}^{k-m} \frac{h_{i-\frac{1}{2}}^j}{j!} \left((u_{(m+j)x})_{i-1}^{h,k} - u_{(m+j)x}(x_{i-1}) \right) \right) + \beta T_i^k.
$$

Let $e_i^k = u_i^k - u(x_i)$ be the error in the kth correction, thus

$$
-\frac{e_{i+1}^k - e_i^k}{h_{i+\frac{1}{2}}} + \frac{e_i^k - e_{i-1}^k}{h_{i-\frac{1}{2}}} + \alpha e_i^k - \alpha e_{i-1}^k + \beta h_i e_i^k = \gamma_i^k - \gamma_{i-1}^k + \delta_i^k \tag{70}
$$

Using the proof of the convergence of the basic solution, first and second corrections and lemma 2.6 together with (48) ,(49), (51) and (53) to get the theorem

Theorem 2.4. *If the unknown solution of (1) belonging to $C^{k+2}(\bar{I})$. Then the error in the kth correction, defined by (70) is, of order $O(h^{k+1})$ in the discrete H_0^1 norm, i.e.*

$$
\left(\sum_{0 \le i \le N} \frac{(e_{i+1}^k - e_i^k)^2}{h_{i+\frac{1}{2}}} \right)^{\frac{1}{2}} \le c h^{k+1} |u|_{k+2,\infty,\bar{I}} \tag{71}
$$

where $e_i^k = u_i^k - u(x_i)$ and $(u_i^k)_i$ are the components of the kth correction defined by (67).

Remark 2.5. *As you have seen, we can generalize the results obtained for general equation $y'' = f(x, y, y')$, where f is a smooth function.*

3. In Two Dimension Space

3.1. **Basic Results.** Considering the second order elliptic problem, with homogeneous boundary conditions

$$\begin{cases} -\Delta u = f, \text{on } \Omega = (0,1)^2 \\ u|_\Gamma = 0. \end{cases} \tag{72}$$

where $\Gamma = \partial\Omega$ is the boundary of Ω and assuming that the solution u is belonging to $C^2(\bar{\Omega})$, and the second member $f \in C(\bar{\Omega})$.

Let $\tau = (K_{ij})_{1\le i\le M;1\le j\le N}$ be an admissible mesh of Ω in the sense of [7], that is satisfying the following assumption

Assumption. *Let M, $N \in \mathbb{N}^*$, $(h_i)_{i=1}^M$, $(k_j)_{j=1}^N$ are positive numbers and such that*

$$\sum_{i=1}^{M} h_i = \sum_{j=1}^{N} k_j = 1,$$

and let $h_0 = h_{M+1} = k_0 = k_{N+1} = 0$. We define:

$$x_{\frac{1}{2}} = 0, \text{ for } i \in \{1, ..., M\} : \ x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + h_i,$$

$$y_{\frac{1}{2}} = 0, \text{ for } j \in \{1, ..., N\} : \ y_{j+\frac{1}{2}} = y_{j-\frac{1}{2}} + k_j,$$

(So that $x_{M+1/2} = y_{N+1/2} = 1$), and

$$Kx_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}],\ Ky_j = [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}],\ K_{ij} = Kx_i \times Ky_j.$$

Let $(x_i)_{i=0}^{M+1}$ and $(y_j)_{j=0}^{N+1}$ be points such that

$$x_{i-\frac{1}{2}} < x_i < x_{i-\frac{1}{2}}, \text{ for } i = 1, ..., M;\ x_0 = 0,\ x_{M+1} = 1,$$

$$y_{j-\frac{1}{2}} < y_j < y_{j-\frac{1}{2}}, \text{ for } j = 1, ..., N;\ y_0 = 0,\ y_{N+1} = 1,$$

and let $x_{i,j} = (x_i, y_j)$ for $i = 1, ..., M$ and $j = 1, ..., N$. Set

$$h_i^- = x_i - x_{i-\frac{1}{2}},\ h_i^+ = x_{i+\frac{1}{2}} - x_i, \text{ for } i = 1, ..., M,\ h_{i+\frac{1}{2}} = x_{i+1} - x_i, \text{ for } i = 0, ..., M,$$

$$k_j^- = y_j - y_{j-\frac{1}{2}},\ k_j^+ = y_{j+\frac{1}{2}} - y_j, \text{ for } j = 1, ..., N,\ k_{j+\frac{1}{2}} = y_{j+1} - y_j, \text{ for } j = 0, ..., N.$$

Assuming that $h_0^+ = h_{M+1}^- = k_0^+ = k_{N+1}^- = 0$ and considering the mesh size $h = \max\{(h_i, i = 1, ..., M), (k_j = 1, ..., N)\}$.

Definition 1. *Let $\mathcal{X}(\tau)$ be the set of functions from Ω to $\mathbb{R}$ piecewise constant over each K_{ij}.*

For $w \in \mathcal{X}(\tau)$, we define the discretes H_0^1-norm and L^2-norm respectively

$$\|w\|_{1,\tau} = \left(\sum_{\substack{i=\overline{0,M} \\ j=\overline{0,N}}} k_j \frac{(w_{i+1,j} - w_{ij})^2}{h_{i+\frac{1}{2}}} + \sum_{\substack{i=\overline{0,M} \\ j=\overline{0,N}}} h_i \frac{(w_{i,j+1} - w_{ij})^2}{k_{j+\frac{1}{2}}} \right)^{\frac{1}{2}} \tag{73}$$

$$\|w\|_{L^2} = \left(\sum_{\substack{i=\overline{0,M} \\ j=\overline{0,N}}} h_i k_j w_{ij}^2 \right)^{1/2} \tag{74}$$

Let $w = (w_{ij})_{0\le i\le M+1, 0\le j\le N+1}$ and Δ^τ be the the following discrete opertor

$$(\Delta^\tau w)_{ij} = -k_j \left(\frac{w_{i+1,j} - w_{ij}}{h_{i+\frac{1}{2}}} - \frac{w_{i,j} - w_{i-1,j}}{h_{i-\frac{1}{2}}} \right) - h_i \left(\frac{w_{i,j+1} - w_{ij}}{k_{j+\frac{1}{2}}} - \frac{w_{i,j} - w_{i,j-1}}{k_{j-\frac{1}{2}}} \right). \tag{75}$$

For a continuous function $g \in C(\Omega)$, we introduce a similar definition:

(76)
$$(\Delta^{*,\tau} g)_{ij} = -k_j\Big(\frac{g(x_{i+1,j}) - g(x_{ij})}{h_{i+\frac{1}{2}}} - \frac{g(x_{i,j} - g(x_{i-1,j})}{h_{i-\frac{1}{2}}}\Big) - h_i\Big(\frac{g(x_{i,j+1}) - g(x_{ij})}{k_{j+\frac{1}{2}}} - \frac{g(x_{i,j}) - g(x_{i,j-1})}{k_{j-\frac{1}{2}}}\Big).$$

To simlify the notations, $\sum_{ij}^1$ denotes $\sum_{\substack{i=\overline{1,M}\\ j=\overline{1,N}}}$ and $\sum_{ij}$ denotes $\sum_{\substack{i=\overline{0,M}\\ j=\overline{0,N}}}$

Integrating the equation (72) over each finite volume K_{ij}, to get

$$-\int_{Ky_j}\Big(u_x(x_{i+\frac{1}{2}}, y) - u_x(x_{i-\frac{1}{2}}, y)\Big)\,dy - \int_{Kx_i}\Big(u_y(x, y_{j+\frac{1}{2}}) - u_y(x, y_{j-\frac{1}{2}})\Big)\,dx = \int_{K_{ij}} f\,dxdy. \tag{77}$$

Taking the first term in left hand side (l.h.s) of (77)

$$\int_{Ky_j}\Big(u_x(x_{i+\frac{1}{2}}, y) - u_x(x_{i-\frac{1}{2}}, y)\Big)\,dy = -k_j\Big(u_x(x_{i+\frac{1}{2}}, y_j) - u_x(x_{i-\frac{1}{2}}, y_j)\Big) - S_{i+\frac{1}{2},j} + S_{i-\frac{1}{2},j},$$

where

$$S_{i+\frac{1}{2},j} = -\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (y - y_j) u_{xy}(x_{i+\frac{1}{2}}, \widehat{y}_j)dy, \tag{78}$$

and $\widehat{y}_j$ is some point lies between y_j and y.

Using, again Taylor's formula, yields

$$u_x(x_{i+\frac{1}{2}}, y_j) = \frac{u(x_{i+1}, y_j) - u(x_i, y_j)}{h_{i+\frac{1}{2}}} + R_{i+\frac{1}{2},j}. \tag{79}$$

Thus the following estimates hold

$$|S_{i+\frac{1}{2},j}| \le c k_j^2 |u_{xy}|_{\infty,\bar{\Omega}}, \text{ and } |R_{i+\frac{1}{2},j}| \le c h_{i+\frac{1}{2}} |u_{2x}|_{\infty,\bar{\Omega}}. \tag{80}$$

By the same way, we can get

$$\begin{aligned} -\int_{Kx_i}\Big(u_y(x, y_{j+\frac{1}{2}}) - u_y(x, y_{j-\frac{1}{2}})\Big)\,dx &= -h_i\left(\frac{u(x_i, y_{j+1}) - u(x_i, y_j)}{k_{j+\frac{1}{2}}} - \frac{u(x_i, y_j) - u(x_i, y_{j-1})}{k_{j+\frac{1}{2}}}\right) \\ &\quad -h_i\Big(R_{i,j+\frac{1}{2}} - R_{i,j-\frac{1}{2}}\Big) - S_{i,j+\frac{1}{2}} + S_{i,j-\frac{1}{2}}, \end{aligned} \tag{81}$$

where

$$|S_{i,j+\frac{1}{2}}| \le c h_i^2 |u_{xy}|_{\infty,\bar{\Omega}}, \text{ and } |R_{i,j+\frac{1}{2}}| \le c k_{j+\frac{1}{2}} |u_{2y}|_{\infty,\bar{\Omega}}. \tag{82}$$

Therefore, the equation (72) becomes after integration as follows

$$(\Delta^{*,\tau} u)_{ij} = \int_{K_{ij}} f\,dxdy + k_j(R_{i+\frac{1}{2},j} - R_{i-\frac{1}{2},j}) + h_i(R_{i,j+\frac{1}{2}} - R_{i,j-\frac{1}{2}}) + S_{i+\frac{1}{2},j} - S_{i-\frac{1}{2},j} + S_{i,j+\frac{1}{2}} - S_{i,j-\frac{1}{2}}.$$

The basic finite volume solution $u^h = (u_{ij})_{i=0,\ldots,M+1, j=0,\ldots,N+1}$ is defined by

$$u_{0j} = u_{M+1,j} = u_{i0} = u_{i,N+1} = 0, \tag{83}$$

and for $(i,j) \in \{1,...,M\} \times \{1,...,N\}$, we have

$$(\Delta^\tau u^h)_{ij} = \int_{K_{ij}} f\,dxdy \tag{84}$$

The existence, uniqueness of the solution u^h, the analysis of the order of the convergence can be justified as done for 1D case (see [7]). More precisely, we have the following theorem

Theorem 3.1. *([7]) If the solution u of the equation (72) belonging to $C^2(\bar{\Omega})$ and $f \in C^2(\bar{\Omega})$. Then the approximate solution $u^h = (u_{ij})$ defined by the boundary condition (83) and the discrete equation (84), satisfies the following estimates*

$$\|e\|_{1,\tau} \leq ch\|u\|_{2,\infty,\bar{\Omega}}, \tag{85}$$

$$\|e\|_{L^2} \leq ch\|u\|_{2,\infty,\bar{\Omega}}, \tag{86}$$

$$\left(\sum_{i,j} k_j h_{i+\frac{1}{2}} e_{ij}^2\right)^{\frac{1}{2}} \leq ch\|u\|_{2,\infty,\bar{\Omega}}. \tag{87}$$

where $e_{ij} = u(x_i, y_j) - u_{ij}$ for $(i,j) \in \{1, ..., M\} \times \{1, ..., N\}$ and vanishes elsewhere.

3.2. **First Correction.** In this section, we assume more regularity for the solution u, i.e. $u \in C^4(\bar{\Omega})$.
Looking, again, at the equation (77), to simplify the notation, let

$$g_i(y) = u_x(x_{i+\frac{1}{2}}, y) - u_x(x_{i-\frac{1}{2}}, y).$$

Using Taylor's formula, we get:

$$-\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} g_i(y)dy = -k_j g_i(y_j) - \frac{{k_j^+}^2 - {k_j^-}^2}{2}(g_i)_y(y_j) - T_{ij}, \tag{88}$$

where

$$\begin{array}{rcl} |T_{ij}| & \leq & ck_j^3|(g_i)_{2y}|_{\infty,K_j} \\ & \leq & ck_j^3 h_i |u_{2x,2y}|_{\infty,\bar{\Omega}}. \end{array} \tag{89}$$

Using again, Taylor's formula, yields

$$\begin{array}{rcl} ({k_j^+}^2 - {k_j^-}^2)(g_i)_y(y_j) & = & ({k_j^+}^2 - {k_j^-}^2)(u_{xy}(x_{i+\frac{1}{2}}, y_j) - u_{xy}(x_{i-\frac{1}{2}}, y_j)) \\ & = & {k_j^+}^2 h_i u_{2x,y}(x_i, y_j) - {k_j^-}^2 h_i u_{2x,y}(x_i, y_{j-1}) + U_{ij}^1 \\ & & +({k_j^+}^2 - {k_j^-}^2)S_{ij}^1 - ({k_j^+}^2 - {k_j^-}^2)S_{ij}^2 \end{array} \tag{90}$$

where

$$|S_{ij}^1| \leq c{h_i^+}^2|u_{3x,y}|_{\infty,\bar{\Omega}} \tag{91}$$

$$|S_{ij}^2| \leq c{h_i^-}^2|u_{3x,y}|_{\infty,\bar{\Omega}} \tag{92}$$

$$|U_{ij}^1| \leq c{k_j^-}^2 h_i k_{j-\frac{1}{2}}|u_{2x,2y}|_{\infty,\bar{\Omega}} \tag{93}$$

By substituting g_i by its value in (88) and by using equality (90), we get

$$
\begin{aligned}
-\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(u_x(x_{i+\frac{1}{2}},y)-u_x(x_{i-\frac{1}{2}},y)\right)dy &= -k_j\left(u_x(x_{i+\frac{1}{2}},y_j)-u_x(x_{i-\frac{1}{2}},y_j)\right)\\
&-h_i\frac{k_j^{+2}}{2}u_{2x,y}(x_i,y_j)+h_i\frac{k_j^{-2}}{2}u_{2x,y}(x_i,y_{j-1})\\
&-\frac{k_j^{+2}-k_j^{-2}}{2}S_{ij}^1-U_{ij}^1+\frac{k_j^{+2}-k_j^{-2}}{2}S_{ij}^2-T_{ij},
\end{aligned}
\tag{94}
$$

but, in the other hand

$$
\frac{u(x_{i+1},y_j)-u(x_i,y_j)}{h_{i+\frac{1}{2}}}=u_x(x_{i+\frac{1}{2}},y_j)+\frac{h_{i+1}^{-}-h_i^{+}}{2}u_{2x}(x_{i+\frac{1}{2}},y_j)+R_{i+\frac{1}{2},j}^2,
\tag{95}
$$

where

$$
|R_{i+\frac{1}{2},j}^2|\le ch_{i+\frac{1}{2}}^2|u_{3x}|_{\infty,\bar{\Omega}}
\tag{96}
$$

Combining (94) and (95) yields that

$$
\begin{aligned}
-\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(u_x(x_{i+\frac{1}{2}},y)-u_x(x_{i-\frac{1}{2}},y)\right)dy &= -k_j\left(\frac{u(x_{i+1},y_j)-u(x_i,y_j)}{h_{i+\frac{1}{2}}}-\frac{u(x_i,y_j)-u(x_{i-1},y_j)}{h_{i-\frac{1}{2}}}\right)\\
&+ k_j\left(\frac{h_{i+1}^{-}-h_i^{+}}{2}u_{2x}(x_{i+\frac{1}{2}},y_j)-\frac{h_i^{-}-h_{i-1}^{+}}{2}u_{2x}(x_{i-\frac{1}{2}},y_j)\right)\\
&- h_i\frac{k_j^{+2}}{2}u_{2x,y}(x_i,y_j)+h_i\frac{k_j^{-2}}{2}u_{2x,y}(x_i,y_{j-1})+k_jR_{i+\frac{1}{2},j}^2\\
&- k_jR_{i-\frac{1}{2},j}^2-\frac{k_j^{+2}-k_j^{-2}}{2}S_{ij}^1+\frac{k_j^{+2}-k_j^{-2}}{2}S_{ij}^2-U_{ij}^1+T_{ij}
\end{aligned}
\tag{97}
$$

and by the same way, we can find similar expansion for the second term in the l.h.s of (77).

$$
\begin{aligned}
-\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(u_y(x,y_{j+\frac{1}{2}})-u_y(x,y_{j-\frac{1}{2}})\right)dy &= -h_i\left(\frac{u(x_i,y_{j+1})-u(x_i,y_j)}{k_{j+\frac{1}{2}}}-\frac{u(x_i,y_j)-u(x_i,y_{j-1})}{k_{j-\frac{1}{2}}}\right)\\
&+ h_i\left(\frac{k_{j+1}^{-}-k_j^{+}}{2}u_{2y}(x_i,y_{j+\frac{1}{2}})-\frac{k_j^{-}-k_{j-1}^{+}}{2}u_{2y}(x_i,y_{j-\frac{1}{2}})\right)\\
&- k_j\frac{h_i^{+2}}{2}u_{x,2y}(x_i,y_j)+k_j\frac{h_i^{-2}}{2}u_{x,2y}(x_{i-1},y_j)+h_iR_{i,j+\frac{1}{2}}^2\\
&- h_iR_{i,j-\frac{1}{2}}^2-\frac{h_i^{+2}-h_i^{-2}}{2}L_{ij}^1+\frac{h_i^{+2}-h_i^{-2}}{2}L_{ij}^2-U_{ij}^2-H_{ij}
\end{aligned}
\tag{98}
$$

where

$$
|R_{i,j+\frac{1}{2}}^2|\le ck_{j+\frac{1}{2}}^2|u_{3y}|_{\infty,\bar{\Omega}}
\tag{99}
$$

$$
|L_{ij}^1|\le ck_j^{+2}|u_{x,3y}|_{\infty,\bar{\Omega}}
\tag{100}
$$

$$
|L_{ij}^2|\le ck_j^{-2}|u_{x,3y}|_{\infty,\bar{\Omega}}
\tag{101}
$$

$$|H_{ij}| \leq ch_i^3 k_j |u_{2x,2y}|_{\infty,\bar{\Omega}} \tag{102}$$

$$|U_{ij}^2| \leq c{h_i^-}^2 k_j h_{i-\frac{1}{2}} |u_{2x,2y}|_{\infty,\bar{\Omega}} \tag{103}$$

Equalities (97) and (98) combined with (77) yields

$$\begin{aligned}
(\Delta^{*,\tau}u)_{ij} \quad = \quad & \int_{K_{ij}} f dxdy - k_j \frac{h_{i+1}^- - h_i^+}{2} u_{2x}(x_{i+\frac{1}{2}}, y_j) + k_j \frac{h_i^- - h_{i-1}^+}{2} u_{2x}(x_{i-\frac{1}{2}}, y_j) \\
& -h_i \frac{k_{j+1}^- - k_j^+}{2} u_{2y}(x_i, y_{j+\frac{1}{2}}) + h_i \frac{k_j^- - k_{j-1}^+}{2} u_{2y}(x_i, y_{j-\frac{1}{2}}) \\
& +h_i \frac{{k_j^+}^2}{2} u_{2x,y}(x_i, y_j) - h_i \frac{{k_j^-}^2}{2} u_{2x,y}(x_i, y_{j-1}) + k_j \frac{{h_i^+}^2}{2} u_{x,2y}(x_i, y_j) - k_j \frac{{h_i^-}^2}{2} u_{x,2y}(x_{i-1}, y_j) \\
& -k_j R_{i+\frac{1}{2},j}^2 + k_j R_{i-\frac{1}{2},j}^2 + \frac{{k_j^+}^2 - {k_j^-}^2}{2} S_{ij}^1 - \frac{{k_j^+}^2 - {k_j^-}^2}{2} S_{ij}^2 + T_{ij} \\
& -h_i R_{i,j+\frac{1}{2}}^2 + h_i R_{i,j-\frac{1}{2}}^2 + \frac{{h_i^+}^2 - {h_i^-}^2}{2} L_{ij}^1 - \frac{{h_i^+}^2 - {h_i^-}^2}{2} L_{ij}^2 + H_{ij} + U_{ij}^1 + U_{ij}^2.
\end{aligned} \tag{104}$$

Let $\Gamma_1 = [0,1] \times \{0\}, \Gamma_2 = \{1\} \times [0,1], \Gamma_3 = [0,1] \times \{1\}, \Gamma_4 = \{0\} \times [0,1]$ be the partial boundaries. The second derivative u_{2x} of u is the solution of the problem

$$\begin{cases} -\Delta v = f_{2x}, \text{on } \Omega = (0,1)^2 \\ v|_{\Gamma_1} = v|_{\Gamma_3} = 0, \\ v|_{\Gamma_2} = -f|_{\Gamma_2}, \\ v|_{\Gamma_4} = -f|_{\Gamma_4} \end{cases} \tag{105}$$

Let $v^h = (v_{ij})$ the finite volume approximation of u_{2x}, then

$$\begin{cases} v_{i0} = v_{i,N+1} = 0, \\ v_{0j} = -f(0, y_j), \\ v_{M+1,j} = -f(1, y_j) \end{cases} \tag{106}$$

and for $(i,j) \in \{1,...,M\} \times \{1,...,N\}$, we have

$$(\Delta^\tau v^h)_{ij} = \int_{K_{ij}} f_{2x} dxdy. \tag{107}$$

Remark 3.1. *We remark that, we have used the same matrix, that used to compute u^h, to compute an optimal approximation for u_{2x}. This implies, in turn, that we have the same order of the convergence as for the solution u provided that u belonging to $C^4(\bar{\Omega})$, i.e. theorem 3.1 holds also for u_{2x} instead of u and v^h instead of u^h.*

Therefore

Lemma 3.1. *Let u be the solution of the equation (72). If $u \in C^4(\bar{\Omega})$. Then the approximation v^h defined by the boundary conditions (106) and the equation (107) satifies the following estimates*

$$\left(\sum_{i,j} h_i k_j (u_{2x}(x_{i+\frac{1}{2}}, y_j) - v_{ij})^2 \right)^{\frac{1}{2}} \leq ch\|u\|_{4,\infty,\bar{\Omega}} \tag{108}$$

(109) $$\left(\sum_{i,j} h_i k_{j+\frac{1}{2}}(u_{2x,y}(x_i, y_j) - \frac{v_{i,j+1} - v_{ij}}{k_{j+\frac{1}{2}}})^2\right)^{\frac{1}{2}} \le ch\|u\|_{4,\infty,\bar{\Omega}}$$

(110) $$\left(\sum_{i,j} h_{i+\frac{1}{2}} k_j(u_{3x}(x_i, y_j) - \frac{v_{i+1,j} - v_{ij}}{h_{i+\frac{1}{2}}})^2\right)^{\frac{1}{2}} \le ch\|u\|_{4,\infty,\bar{\Omega}}$$

Proof
1-Using triangular inequality combined with estimate (86) of theorem 3.1 yield

$$\begin{aligned}\left(\sum_{i,j} h_i k_j (u_{2x}(x_{i+\frac{1}{2}}, y_j) - v_{ij})^2\right)^{\frac{1}{2}} &\le \left(\sum_{i,j} h_i k_j (u_{2x}(x_{i+\frac{1}{2}}, y_j) - u_{2x}(x_i, y_j)^2\right)^{\frac{1}{2}} \\ &+ \left(\sum_{i,j} h_i k_j (u_{2x}(x_i, y_j) - v_{ij})^2\right)^{\frac{1}{2}} \\ &\le ch\left(\|u_{3x}\|_\infty + \|u\|_{4,\infty,\bar{\Omega}}\right) \\ &\le ch\|u\|_{4,\infty,\bar{\Omega}}.\end{aligned}$$

2-Estimates (109) and (110) can be proven by the same way, i.e. we use triangular inequality and theorem 3.1.□
Let $w^h = (w_{ij})$ be a discrete function, to simplify the natation, we define the following discrete operators

$$\partial_1^x w_{ij} = \frac{w_{i+1,j} - w_{ij}}{h_{i+\frac{1}{2}}}, \text{ and } \partial_1^y w_{ij} = \frac{w_{i,j+1} - w_{ij}}{k_{j+\frac{1}{2}}}.$$

Now, we are able to define a new approximation $u_1^h = (u_{ij}^1)$, called correction, of order $O(h^2)$ (as we will see) defined by the boundary conditions (83) and the following discrete equation for $(i,j) \in \{1, ..., M\} \times \{1, ..., N\}$

(111) $$(\Delta^\tau u_1^h)_{ij} = \int_{K_{ij}} f\,dx\,dy + \left(\gamma_{i+\frac{1}{2},j} - \gamma_{i-\frac{1}{2},j}\right) + \left(\gamma_{i,j+\frac{1}{2}} - \gamma_{i,j-\frac{1}{2}}\right) + \delta_{ij},$$

where

(112) $$\gamma_{i+\frac{1}{2},j} = -k_j \frac{h_{i+1}^- - h_i^+}{2} v_{ij},$$

(113) $$\gamma_{i,j+\frac{1}{2}} = -h_i \frac{k_{j+1}^- - k_j^+}{2}(-f(x_i, y_{j+\frac{1}{2}}) - v_{ij}),$$

(114) $$\left\{\begin{aligned} \delta_{ij} &= h_i \frac{{k_j^+}^2}{2}\partial_1^y v_{ij} - h_i \frac{{k_j^-}^2}{2}\partial_1^y v_{i,j-1} + k_j \frac{{h_i^+}^2}{2}(-f_x(x_i, y_j) - \partial_1^x v_{ij}) \\ &\quad -k_j \frac{{h_i^-}^2}{2}(-f_x(x_{i-1}, y_j) - \partial_1^x v_{i-1,j}).\end{aligned}\right.$$

Remark 3.2. *As you can see that, we have used only tha approximation of u_{2x} and the fact that $u_{2y} = -f - u_{2x}$ to approximate all the higher pointwise derivatives in the r.h.s of (104).*

To analyse the error, let $e^1_{ij} = u^1_{ij} - u(x_i, y_j)$ be the error in the first correction, by substracting (104) from (111), we get

$$(\Delta^\tau e^1)_{ij} = \bar{\gamma}_{i+\frac{1}{2},j} - \bar{\gamma}_{i-\frac{1}{2},j} + \bar{\gamma}_{i,j+\frac{1}{2}} - \bar{\gamma}_{i,j-\frac{1}{2}} + \bar{\delta}_{ij}, \tag{115}$$

where

$$\bar{\gamma}_{i+\frac{1}{2},j} = -k_j \frac{h^-_{i+1} - h^+_i}{2}(v_{ij} - u_{2x}(x_{i+\frac{1}{2}}, y_j)) + k_j R^2_{i+\frac{1}{2},j}, \tag{116}$$

$$\bar{\gamma}_{i,j+\frac{1}{2}} = -h_i \frac{k^-_{j+1} - k^+_j}{2}(u_{2x}(x_i, y_{j+\frac{1}{2}}) - v_{ij}) + h_i R^2_{i,j+\frac{1}{2}}, \tag{117}$$

$$\begin{aligned}\bar{\delta}_{ij} = h_i \frac{k^{+2}_j}{2}(\partial^y_1 v_{ij} - u_{2x,y}(x_{i,j})) - h_i \frac{k^{-2}_j}{2}(\partial^y_1 v_{i,j-1} - u_{2x,y}(x_{i,j-1})) - h_i \frac{k^{-2}_j}{2}(\partial^y_1 v_{i,j-1} - u_{2x,y}(x_{i,j-1}))\\ +k_j \frac{h^{+2}_i}{2}(u_{3x}(x_i, y_j) - \partial^x_1 v_{ij}) - k_j \frac{h^{-2}_i}{2}(u_{3x}(x_{i-1}, y_j) - \partial^x_1 v_{i-1,j})\\ -\frac{k^{+2}_j - k^{-2}_j}{2} S^1_{ij} + \frac{k^{+2}_j - k^{-2}_j}{2} S^2_{ij} \frac{h^{+2}_i - h^{-2}_i}{2} L^1_{ij} + \frac{h^{+2}_i - h^{-2}_i}{2} L^2_{ij}\\ -T_{ij} - H_{ij} - U^1_{ij} - U^2_{ij}\end{aligned} \tag{118}$$

3.3. **Convergence Order of the First Correction.** Multiplying both sides of (115) by e^1_{ij} and summing over $(i,j) \in \{1, ..., M\} \times \{1, ..., N\}$, to get

$$\begin{aligned}-\sum_{i,j}{}^1 k_j \frac{e^1_{i+1,j} - e^1_{ij}}{h_{i+\frac{1}{2}}} e^1_{ij} + \sum_{i,j}{}^1 k_j \frac{e^1_{ij} - e^1_{i-1,j}}{h_{i-\frac{1}{2}}} e^1_{ij} - \sum_{i,j}{}^1 h_i \frac{e^1_{i,j+1} - e^1_{ij}}{k_{j+\frac{1}{2}}} e^1_{ij} + \sum_{i,j}{}^1 h_i \frac{e^1_{ij} - e^1_{i,j-1}}{k_{j-\frac{1}{2}}} e_{ij}\\ = \sum_{i,j}{}^1 \bar{\gamma}_{i+\frac{1}{2},j} e^1_{ij} - \sum_{i,j}{}^1 \bar{\gamma}_{i-\frac{1}{2},j} e^1_{ij} + \sum_{i,j}{}^1 \bar{\gamma}_{i,j+\frac{1}{2}} e^1_{ij} - \sum_{i,j}{}^1 \bar{\gamma}_{i,j-\frac{1}{2}} e^1_{ij} + \sum_{i,j}{}^1 \bar{\delta}_{ij} e^1_{ij}\end{aligned} \tag{119}$$

Reordering equation (119) and using the fact that e^1_{ij} vanishs on the boundary mesh points, to get

$$\|e^1\|^2_{1,\tau} = -\sum_{i,j} \bar{\gamma}_{i+\frac{1}{2},j}(e^1_{i+1,j} - e^1_{ij}) - \sum_{i,j} \bar{\gamma}_{i,j+\frac{1}{2}}(e^1_{i,j+1} - e^1_{ij}) + \sum_{i,j}{}^1 \bar{\delta}_{ij} e^1_{ij} \tag{120}$$

We should now estimate each term in the r.h.s of (120). The first and the second term can be handled by the same way. Hence, it is suffices to estimate the first and the last ones. Using Cauchy-Schwarz inequality yields that

$$\begin{aligned}|\sum_{i,j} \bar{\gamma}_{i+\frac{1}{2},j}(e^1_{i+1,j} - e^1_{ij})| &\leq (\sum_{i,j} \frac{\bar{\gamma}^2_{i+\frac{1}{2},j} h_{i+\frac{1}{2}}}{k_j})^{\frac{1}{2}} (\sum_{i,j} k_j \frac{(e^1_{i+1,j} - e^1_{ij})^2}{h_{i+\frac{1}{2}}})^{\frac{1}{2}}\\ &\leq (\sum_{i,j} \frac{\bar{\gamma}^2_{i+\frac{1}{2},j} h_{i+\frac{1}{2}}}{k_j})^{\frac{1}{2}} \|e^1\|_{1,\tau}\end{aligned} \tag{121}$$

Using triangular inequality, in order to get

$$(\sum_{i,j} \frac{\bar{\gamma}^2_{i+\frac{1}{2},j} h_{i+\frac{1}{2}}}{k_j})^{\frac{1}{2}} \leq \left(\sum_{i,j} \frac{k^2_j (h^-_{i+1} - h^+_i)^2}{4k_j} h_{i+\frac{1}{2}} (v_{ij} - u_{2x}(x_{i+\frac{1}{2}}, y_j))^2\right)^{\frac{1}{2}} + \left(\sum_{i,j} \frac{k^2_j (R^2_{i+\frac{1}{2}})^2}{k_j} h_{i+\frac{1}{2}}\right)^{\frac{1}{2}}$$

Using estimate (108) of lemma 3.1 (combined with estimate (87) of theorem 3.1) and inequality (96), to get

$$
\begin{aligned}
\left(\sum_{i,j}\frac{\bar{\gamma}^2_{i+\frac{1}{2},j}h_{i+\frac{1}{2}}}{k_j}\right)^{\frac{1}{2}} &\leq c\left(h^2\|u\|_{4,\infty,\bar{\Omega}}+h^2\|u\|_{3,\infty,\bar{\Omega}}\right)\\
&\leq ch^2\|u\|_{4,\infty,\bar{\Omega}} \qquad (122)
\end{aligned}
$$

Comming back now to the last term in the r.h.s of (120), using triangular inequality yields that

$$
\begin{aligned}
|\sum_{i,j}^{1}\bar{\delta}_{ij}e^1_{ij}| &\leq \sum_{i,j}^{1}h_i\frac{k_j^{+2}}{2}|\partial_1^y v_{ij}-u_{2x,y}(x_i,y_j)||e^1_{ij}|+\sum_{i,j}^{1}h_i\frac{k_j^{-2}}{2}|\partial_1^y v_{i,j-1}-u_{2x,y}(x_i,y_{j-1})||e^1_{ij}|\\
&+\sum_{i,j}^{1}k_j\frac{h_i^{+2}}{2}|u_{3x}(x_i,y_j)-\partial_1^x v_{ij}||e^1_{ij}|+\sum_{i,j}^{1}k_j\frac{h_i^{-2}}{2}|u_{3x}(x_{i-1},y_j)-\partial_1^x v_{i-1,j}||e^1_{ij}|\\
&+\sum_{i,j}^{1}\frac{|k_j^{+2}-k_j^{-2}|}{2}|S^1_{ij}||e^1_{ij}|+\sum_{i,j}^{1}\frac{|k_j^{+2}-k_j^{-2}|}{2}|S^2_{ij}||e^1_{ij}|+\sum_{i,j}^{1}|T_{ij}||e^1_{ij}|\\
&+\sum_{i,j}^{1}\frac{|h_i^{+2}-h_i^{-2}|}{2}|L^1_{ij}||e^1_{ij}|+\sum_{i,j}^{1}\frac{|h_i^{+2}-h_i^{-2}|}{2}|L^2_{ij}||e^1_{ij}|+\sum_{i,j}^{1}|H_{ij}||e^1_{ij}|\\
&+\sum_{i,j}^{1}|U^1_{ij}||e^1_{ij}|+\sum_{i,j}^{1}|U^2_{ij}||e^1_{ij}| \qquad (123)
\end{aligned}
$$

Begininig by the first term in the r.h.s of (123), using the Cauchy-Schwars inequality and the estimate (109) of lemma 3.1

$$
\begin{aligned}
\sum_{i,j}^{1}h_i\frac{k_j^{+2}}{2}|\partial_1^y v_{ij}-u_{2x,y}(x_i,y_j)||e^1_{ij}| &\leq \sum_{i,j}^{1}h_i\frac{k_{j+\frac{1}{2}}^{\;2}}{2}|\partial_1^y v_{ij}-u_{2x,y}(x_i,y_j)||e^1_{ij}|\\
&\leq ch\left(\sum_{i,j}^{1}h_ik_{j+\frac{1}{2}}(\partial_1^y v_{ij}-u_{2x,y}(x_i,y_j))^2\right)^{\frac{1}{2}}\left(\sum_{i,j}^{1}h_ik_{j+\frac{1}{2}}(e^1_{ij})^2\right)^{\frac{1}{2}}\\
(124)\qquad &\leq ch^2\|u\|_{4,\infty,\bar{\Omega}}\left(\sum_{i,j}^{1}h_ik_{j+\frac{1}{2}}(e^1_{ij})^2\right)^{\frac{1}{2}}
\end{aligned}
$$

This implies that

$$
\sum_{i,j}^{1}h_i\frac{k_j^{+2}}{2}|\partial_1^y v_{ij}-u_{2x,y}(x_i,y_j)||e^1_{ij}|\leq ch^2\|e^1\|_{1,\tau}\|u\|_{4,\infty,\bar{\Omega}} \qquad (125)
$$

and by the same way, we can handle the second term in the r.h.s of (123). Indeed

$$
\begin{aligned}
\sum_{i,j}^{1} h_i \frac{k_j^{-2}}{2} |\partial_1^y v_{i,j-1} - u_{2x,y}(x_i, y_{j-1})||e_{ij}^1| &\leq ch \sum_{i,j}^{1} h_i k_j^{-} |\partial_1^y v_{i,j-1} - u_{2x,y}(x_i, y_{j-1})||e_{ij}^1| \\
&\leq ch (\sum_{i,j}^{1} h_i k_j^{-} (e_{ij}^1)^2)^{\frac{1}{2}} (\sum_{i,j}^{1} h_i k_j^{-} (\partial_1^y v_{i,j-1} - u_{2x,y}(x_i, y_{j-1}))^2)^{\frac{1}{2}} \\
&\leq ch \|e^1\|_{L^2} \left(\sum_{i,j}^{1} h_i k_{j-\frac{1}{2}} (\partial_1^y v_{i,j-1} - u_{2x,y}(x_i, y_{j-1}))^2 \right)^{\frac{1}{2}} \\
&\leq ch \|e^1\|_{L^2} \left(\sum_{\substack{i=\overline{1,M} \\ j=\overline{0,N}}} h_i k_{j+\frac{1}{2}} (\partial_1^y v_{ij} - u_{2x,y}(x_i, y_j))^2 \right)^{\frac{1}{2}} .
\end{aligned}
$$

This with estimate (109) imply that

$$
\sum_{i,j}^{1} h_i \frac{k_j^{-2}}{2} |\partial_1^y v_{i,j-1} - u_{2x,y}(x_i, y_{j-1})||e_{ij}^1| \leq ch^2 \|e^1\|_{1,\tau} \|u\|_{4,\infty,\bar{\Omega}} \tag{126}
$$

Taking, now, a look at the other kind of terms. Using estimate (91) to get

$$
\begin{aligned}
\sum_{i,j}^{1} \frac{|k_j^{+2} - k_j^{-2}|}{2} |S_{ij}^1||e_{ij}^1| &\leq c\|u\|_{4,\infty,\bar{\Omega}} \sum_{i,j}^{1} k_j^2 h_i^2 |e_{ij}^1| \\
&\leq ch^2 \|u\|_{4,\infty,\bar{\Omega}} \sum_{i,j}^{1} k_j h_i |e_{ij}^1| \\
&\leq ch^2 \|u\|_{4,\infty,\bar{\Omega}} \left(\sum_{i,j}^{1} k_j h_i \right)^{\frac{1}{2}} \left(\sum_{i,j}^{1} k_j h_i (e_{ij}^1)^2 \right)^{\frac{1}{2}} .
\end{aligned}
$$

Hence

$$
\sum_{i,j}^{1} \frac{|k_j^{+2} - k_j^{-2}|}{2} |S_{ij}^1||e_{ij}^1| \leq ch^2 \|u\|_{4,\infty,\bar{\Omega}} \|e^1\|_{1,\tau}. \tag{127}
$$

By the same way, we can find the same estimate for terms corresponding to S_{ij}^2, L_{ij}^1 , L_{ij}^2, T_{ij}, U_{ij}^1 and U_{ij}^2 .
Inequalities (123)-(127) yield that

$$
|\sum_{i,j}^{1} \bar{\delta}_{ij} e_{ij}^1| \leq ch^2 \|u\|_{4,\infty,\bar{\Omega}} \|e^1\|_{1,\tau}. \tag{128}
$$

Combining equality (120) with inequalities (122) and (128) yields

Theorem 3.2. *If the solution u of the equation (72) belonging to $C^4(\bar{\Omega})$. Let u^h be the basic finite volume solution of boundary condition (83) and discrete equation (84). Then the finite volume approximation $u_1^h = (u_{ij}^1)$ defined by the boundary condition (83) and the discrete equation (111), satisfies the following $O(h)$ improvement in H_0^1-norm*

$$
\|e^1\|_{1,\tau} \leq ch^2 \|u\|_{4,\infty,\bar{\Omega}}, \tag{129}
$$

$$\left(\sum_{i,j} k_j h_{i+\frac{1}{2}} (e^1_{ij})^2\right)^{\frac{1}{2}} \leq ch^2 \|u\|_{4,\infty,\bar{\Omega}}, \tag{130}$$

$$\|e^1\|_{L^2} \leq ch^2 \|u\|_{4,\infty,\bar{\Omega}}, \tag{131}$$

where $e^1_{ij} = u(x_i, y_j) - u^1_{ij}$ *for* $(i,j) \in \{1, ..., M\} \times \{1, ..., N\}$ *and vanishes elsewhere.*

Remark 3.3. *Numerical results schows that the order of the convergence in* L^2 *norm even of the basic solution* u^h *is* $O(h^2)$*, i.e. the order in (85) and (86) is* $O(h^2)$ *but the coeffecient of the order in (130) and (131) are smaller than to those ones of (86) and (87).*

3.4. **The Second Correction and Higher Order of Corrections.** In this subsection, we give an idea allowing us to construct second correction,i.e.the order of the convergence is $O(h^3)$, this result can be extended to construct an arbitrary correction we wish. To compute the second correction, we use the first correction to estimate some pointwise derivatives of the solution u and the fact the equation (72) satisfying by u. It suffices to remark that, provided that at least $u \in C^4(\bar{\Omega})$

$$\begin{aligned} -\int_{K_{ij}} f dxdy &= -k_j \left(u_x(x_{i+\frac{1}{2}}, y_j) - u_x(x_{i-\frac{1}{2}}, y_j)\right) - h_i \left(u_y(x_i, y_{j+\frac{1}{2}}) - u_y(x_i, y_{j-\frac{1}{2}})\right) \\ &\quad - h_i \frac{{k_j^+}^2 - {k_j^-}^2}{2} u_{2x,y}(x_i, y_j) - k_j \frac{{h_i^+}^2 - {h_i^-}^2}{2} u_{x,2y}(x_i, y_j) \\ &\quad - \frac{{h_i^+}^2 - {h_i^-}^2}{2} k_j \frac{{h_i^+}^2 - {h_i^-}^2}{2} f_{xy}(x_i, y_j) + O(k_j^4 h_i) - O(h_i^4 k_j) + O(k_j^2 h_i^3) - O(k_j^3 h_i^2). \end{aligned}$$

To approximate the pointwise derivative $u_{2x,y}(x_i, y_j)$, we compute the first correction to the unknown solution u_{2x} (because it is satifying the same equation that is satisfying by u, i.e. 105, this first correction is of order $O(h^2)$ in H^1_0, this means that $u_{2x,y}(x_i, y_j)$ can be approximated by an $O(h^2)$. By similar way, we can approximate $u_{2x,y}(x_i, y_j)$.

Comming back to $u_x(x_{i+\frac{1}{2}}, y_j) - u_x(x_{i-\frac{1}{2}}, y_j)$, the derivatives will appeared here in the approximation of $u_x(x_{i+\frac{1}{2}}, y_j)$ and $u_x(x_{i-\frac{1}{2}}, y_j)$ are similars to those obtained in one dimensional space for the second correction, and consequently, we can use the approximations just obtained to u_{2x} to approximate such derivatives.

3.5. **Some Extensions of the Results.** So far, we have considered the Laplace model, where the second derivative of the solution u are also solutions of the same equation. In this section, we attempt to extend results obtained, to some second order elliptic problems, where the second derivatives of the solution u are also solutions but for second member depends on the solution u itself, its derivatives and a given function.

The idea will be used is to approximate these terms by theirs ones corresponding in the finite volume solution, i.e. u and derivatives of u will be replaced by u^h and divided difference of u^h respectively.

Let us consider the following model

$$\begin{cases} -\Delta u + pu = f, \text{on } \Omega = (0,1)^2 \\ u|_\Gamma = 0. \end{cases} \tag{132}$$

where p is a given function and $p \geq 0$.

We use the same scheme that used for Laplace model. As done above, we look, at first, for the finite volume solution u^h (basic solution), after, we look for a convenient expansion for the error, where we try to approximate the derivatives of the unknown solution u by using the basic solution u^h.

3.6. **The Finite Volume Approximation (Basic Solution).** We use the same notations that used in the second section, therefore, for $f, p \in C^1(\bar{\Omega})$

$$\begin{aligned}(\Delta^{*,\tau} u)_{ij} + h_i k_j p_{ij} u(x_i, y_j) &= \int_{K_{ij}} f dx dy + k_j(R_{i+\frac{1}{2},j} - R_{i-\frac{1}{2},j}) + h_i(R_{i,j+\frac{1}{2}} - R_{i,j-\frac{1}{2}}) \\ &+ S_{i+\frac{1}{2},j} - S_{i-\frac{1}{2},j} + S_{i,j+\frac{1}{2}} - S_{i,j-\frac{1}{2}} - N_{ij},\end{aligned} \tag{133}$$

where $p_{ij} = p(x_i, y_j)$, $R_{i+\frac{1}{2},j}$, $R_{i,j+\frac{1}{2}}$, $S_{i,j+\frac{1}{2}}$ are defined as in (78),(79), (80), (80), (82) and (82). N_{ij} is defined by

$$N_{ij} = \int_{K_{ij}} \left((x - x_i) \frac{\partial(pu)}{\partial x}(\widehat{a}_{ij}) + (y - y_j) \frac{\partial(pu)}{\partial y}(\widehat{a}_{ij}) \right) dx dy \tag{134}$$

where $\widehat{a}_{ij}$ is a some point in K_{ij}. Then the following estimation holds

$$|N_{ij}| \le c h_i k_j (h_i + k_j) \|u\|_{1,\infty,\bar{\Omega}}. \tag{135}$$

The basic solution $u^h = (u_{ij})_{i=0,\ldots,M+1, j=0,\ldots,N+1}$, which will approximate the solution u of the equation (132) is defined by

$$u_{0j} = u_{M+1,j} = u_{i0} = u_{i,N+1} = 0, \tag{136}$$

and for $(i, j) \in \{1, ..., M\} \times \{1, ..., N\}$, we have

$$(\Delta^\tau u^h)_{ij} + h_i k_j p_{ij} u_{ij} = \int_{K_{ij}} f \, dx \, dy. \tag{137}$$

The existence and uniqueness can be done by using the same techniques in 1D (see [7]).
Using techniques that used in the second section yields (given in [7])

Theorem 3.3. *If the solution u of the equation (132) belonging to $C^2(\bar{\Omega})$,the coefficient p belonging to $C^1(\bar{\Omega})$ and $f \in C(\bar{\Omega})$. Then the approximate solution $u^h = (u_{ij})$ defined by the boundary condition (136) and the discrete equation (137), satisfies the following estimates*

$$\|e\|_{1,\tau} \le ch\|u\|_{2,\infty,\bar{\Omega}}, \tag{138}$$

$$\left(\sum_{i,j} k_j h_{i+\frac{1}{2}} e_{ij}^2 \right)^{\frac{1}{2}} \le ch\|u\|_{2,\infty,\bar{\Omega}}, \tag{139}$$

$$\|e\|_{L^2} \le ch\|u\|_{2,\infty,\bar{\Omega}}, \tag{140}$$

where $e_{ij} = u(x_i, y_j) - u_{ij}$ for $(i, j) \in \{1, ..., M\} \times \{1, ..., N\}$ and vanishes elsewhere.

3.7. **The First Correction.** We proceed as in the third section, we begin by finding an expansion of the error. Begining by $\int_{K_{ij}} pudxdy$
We have

$$\int_{K_{ij}} pudxdy = h_ik_jp_{ij}u(x_i,y_j) + k_j\frac{h_i^{+2}-h_i^{-2}}{2}\frac{\partial(pu)}{\partial x}(a_{ij}) + h_i\frac{k_j^{+2}-k_j^{-2}}{2}\frac{\partial(pu)}{\partial y}(a_{ij}) + N_{ij}^1, \tag{141}$$

where $a_{ij} = (x_i, y_j)$ and

$$|N_{ij}^1| \leq ch_ik_jh^2\|u\|_{2,\infty,\bar{\Omega}}. \tag{142}$$

Combining equalities (104) and (141) yields
(143)

$$\begin{aligned}
(\quad \Delta^{*,\tau}u)_{ij}+h_ik_jp_{ij}u(x_i,y_j) &= \int_{K_{ij}} fdxdy\\
&-k_j\tfrac{h_{i+1}^{-}-h_i^{+}}{2}u_{2x}(x_{i+\frac12},y_j) + k_j\tfrac{h_i^{-}-h_{i-1}^{+}}{2}u_{2x}(x_{i-\frac12},y_j) - h_i\tfrac{k_{j+1}^{-}-k_j^{+}}{2}u_{2y}(x_i,y_{j+\frac12}) + h_i\tfrac{k_j^{-}-k_{j-1}^{+}}{2}u_{2y}(x_i,y_{j-\frac12})\\
&+h_i\tfrac{k_j^{+2}}{2}u_{2x,y}(x_i,y_j) - h_i\tfrac{k_j^{-2}}{2}u_{2x,y}(x_i,y_{j-1}) + k_j\tfrac{h_i^{+2}}{2}u_{x,2y}(x_i,y_j) - k_j\tfrac{h_i^{-2}}{2}u_{x,2y}(x_{i-1},y_j)\\
&-k_j\tfrac{h_i^{+2}-h_i^{-2}}{2}\left(p_x(x_i,y_j)u(x_i,y_j)+p_{ij}u_x(x_i,y_j)\right) - h_i\tfrac{k_j^{+2}-k_j^{-2}}{2}\left(p_y(x_i,y_j)u(x_i,y_j)+p_{ij}u_y(x_i,y_j)\right)\\
&-k_j(R^2_{i+\frac12,j}-R^2_{i-\frac12,j}) + \tfrac{k_j^{+2}-k_j^{-2}}{2}(S^1_{ij}-S^2_{ij}) - h_i(R^2_{i,j+\frac12}+R^2_{i,j-\frac12}) + \tfrac{h_i^{+2}-h_i^{-2}}{2}(L^1_{ij}-L^2_{ij})\\
&+T_{ij}+H_{ij}+U^1_{ij}+U^2_{ij}-N^1_{ij}.
\end{aligned}$$

We look, now, for an approximation to the second derivative u_{2x} of u by using the same matrix that used to compute the basic solution u^h. Remarking that u_{2x} is the solution of the following equation

$$-\Delta v + pv = f_{2x} - p_{2x}u - 2p_xu_x \tag{144}$$

with the boundary conditions

$$\begin{cases} v|_{\Gamma_1} = v|_{\Gamma_3} = 0,\\ v|_{\Gamma_2} = -f|_{\Gamma_2},\\ v|_{\Gamma_4} = -f|_{\Gamma_4}.\end{cases} \tag{145}$$

Hence an approximation $v^h = (v_{ij})$ to u_{2x} can be defined as

$$\begin{cases} v_{i0} = v_{i,N+1} = 0,\\ v_{0j} = -f(0,y_j),\\ v_{M+1,j} = -f(1,y_j)\end{cases} \tag{146}$$

and for $(i,j) \in \{1,...,M\}\times\{1,...,N\}$, we have

$$\begin{aligned}
(\Delta^\tau v^h)_{ij} \quad &+ \quad h_ik_jp_{ij}v_{ij} = \int_{K_{ij}} f_{2x}dxdy\\
&- \quad h_ik_jp_{2x}(x_i,y_j)u_{ij} - 2h_i^{+}k_jp_x(x_i,y_j)\partial_1^x u_{ij} - 2h_i^{-}k_jp_x(x_i,y_j)\partial_1^x u_{i-1,j}
\end{aligned} \tag{147}$$

where u_{ij} are the components of the finite volume solution u^h defined by (136)-(137).
To analyse the convergence of the finite volume approximation v^h, let $v = u_{2x}$ and using equality

(133) to get

$$
\begin{aligned}
(\Delta^{*,\tau} u_{2x})_{ij} &+ h_i k_j v(x_i, y_j) = \int_{K_{ij}} (f - p_{2x} u - 2 p_x u_x) dx dy \\
&+ k_j \left(R_{i+\frac{1}{2},j}(u_{2x}) - R_{i-\frac{1}{2},j}(u_{2x}) \right) + h_i \left(R_{i,j+\frac{1}{2}}(u_{2x}) - R_{i,j-\frac{1}{2}}(u_{2x}) \right) \\
&+ S_{i+\frac{1}{2},j}(u_{2x}) - S_{i-\frac{1}{2},j}(u_{2x}) + S_{i,j+\frac{1}{2}}(u_{2x}) - S_{i,j-\frac{1}{2}}(u_{2x}) - N_{ij}(u_{2x}),
\end{aligned}
\tag{148}
$$

where $R_{i+\frac{1}{2},j}(u_{2x})$, $R_{i,j+\frac{1}{2}}(u_{2x})$, $S_{i+\frac{1}{2},j}(u_{2x})$, $S_{i,j+\frac{1}{2}}(u_{2x})$, $N_{ij}(u_{2x})$ are the same previous expansions by substituting each u by u_{2x}.
This implies that

$$
\begin{aligned}
(\Delta^{*,\tau} u_{2x})_{ij} &+ h_i k_j v(x_i, y_j) = \int_{K_{ij}} f dx dy \\
&- h_i k_j p_{2x}(x_i, y_j) u(x_i, y_j) - 2 k_j h_i^+ p_x(x_i, y_j) u_x(x_i, y_j) \\
&- 2 k_j h_i^- p_x(x_i, y_j) u_x(x_{i-1}, y_j) + O(h_i k_j h) + O(h_i^- k_j h_{i-\frac{1}{2}}) \\
&+ k_j R_{i+\frac{1}{2},j}(u_{2x}) - k_j R_{i-\frac{1}{2},j}(u_{2x}) + h_i R_{i,j+\frac{1}{2}}(u_{2x}) - h_i R_{i,j-\frac{1}{2}}(u_{2x}) \\
&+ S_{i+\frac{1}{2},j}(u_{2x}) - S_{i-\frac{1}{2},j}(u_{2x}) + S_{i,j+\frac{1}{2}}(u_{2x}) - S_{i,j-\frac{1}{2}}(u_{2x}) - N_{ij}(u_{2x})
\end{aligned}
\tag{149}
$$

Let $r^h = (r_{ij})_{i,j} = (v(x_i, y_j) - v_{ij})_{i,j}$ be the error in the approxomation (147). Substracting (147) from (149), we get for $(i,j) \in \{1, ..., M\} \times \{1, ..., N\}$

$$
(\Delta^\tau r)_{ij} + h_i k_j p_{ij} r_{ij} = \alpha_{i+\frac{1}{2},j} - \alpha_{i-\frac{1}{2},j} + \alpha_{i,j+\frac{1}{2}} - \alpha_{i,j-\frac{1}{2}} + \delta_{ij} \tag{150}
$$

where

$$
\alpha_{i+\frac{1}{2},j} = k_j R_{i+\frac{1}{2},j}(u_{2x}) + S_{i+\frac{1}{2},j}(u_{2x}), \tag{151}
$$

$$
\alpha_{i,j+\frac{1}{2}} = h_i R_{i,j+\frac{1}{2}}(u_{2x}) + S_{i,j+\frac{1}{2}}(u_{2x}), \tag{152}
$$

$$
\left\{
\begin{aligned}
\delta_{ij} &= -h_i k_j p_{2x}(x_i, y_j)(u(x_i, y_j) - u_{ij}) - 2 k_j h_i^+ p_x(x_i, y_j)(u_x(x_i, y_j) - \partial_1^x u_{ij}) \\
&\quad -2 k_j h_i^- p_x(x_i, y_j)(u_x(x_{i-1}, y_j) - \partial_1^x u_{i-1,j}) + O(h_i k_j h) + O(h_i^- k_j h_{i-\frac{1}{2}}).
\end{aligned}
\right.
\tag{153}
$$

Multiplying both sides of (150) by r_{ij} and summing over $(i,j) \in \{1, ..., M\} \times \{1, ..., N\}$, to get

$$
\|r^h\|_{1,\tau}^2 + \sum_{i,j}^1 h_i k_j p_{ij} r_{ij}^2 = - \sum_{i,j} \alpha_{i+\frac{1}{2},j}(r_{i+1,j} - r_{ij}) - \sum_{i,j} \alpha_{i+\frac{1}{2},j}(r_{i,j+1} - r_{ij}) + \sum_{i,j}^1 \delta_{ij} r_{ij}. \tag{154}
$$

Using the tricks those used to bound the error in the first correction (subsection 3.3) to obtain the following optimal approximation to u_{2x}

Lemma 3.2. *Let u be the solution of the equation (132). If $u \in C^4(\bar{\Omega})$ and $p \in C^2(\bar{\Omega})$. Then the approximation v^h defined by the boundary conditions (146) and the equation (147) satifies the following estimates*

$$
\left(\sum_{i,j} h_i k_j (u_{2x}(x_{i+\frac{1}{2}}, y_j) - v_{ij})^2 \right)^{\frac{1}{2}} \leq ch \|u\|_{4,\infty,\bar{\Omega}} \tag{155}
$$

$$
\left(\sum_{i,j} h_{i+\frac{1}{2}} k_j (u_{2x}(x_{i+\frac{1}{2}}, y_j) - v_{ij})^2 \right)^{\frac{1}{2}} \leq ch \|u\|_{4,\infty,\bar{\Omega}} \tag{156}
$$

$$\left(\sum_{i,j} k_{j+\frac{1}{2}} h_i (u_{2x}(x_{i+\frac{1}{2}}, y_j) - v_{ij})^2\right)^{\frac{1}{2}} \leq ch\|u\|_{4,\infty,\bar{\Omega}} \tag{157}$$

$$\left(\sum_{i,j} h_i k_{j+\frac{1}{2}} (u_{2x,y}(x_i, y_j) - \frac{v_{i,j+1} - v_{ij}}{k_{j+\frac{1}{2}}})^2\right)^{\frac{1}{2}} \leq ch\|u\|_{4,\infty,\bar{\Omega}} \tag{158}$$

$$\left(\sum_{i,j} h_{i+\frac{1}{2}} k_j (u_{3x}(x_i, y_j) - \frac{v_{i+1,j} - v_{ij}}{h_{i+\frac{1}{2}}})^2\right)^{\frac{1}{2}} \leq ch\|u\|_{4,\infty,\bar{\Omega}} \tag{159}$$

After having acheived an optimal approximation for u_{2x} , we have to find an improvement of the basic solution u^h, i.e. correction of order $O(h^2)$. Looking again at the equality (143) and rewrite

$$\begin{aligned}
(\Delta^{*,\tau} u)_{ij} \quad + \quad & h_i k_j p_{ij} u(x_i, y_j) = \int_{K_{ij}} f dx dy - k_j \frac{h_{i+1}^- - h_i^+}{2} u_{2x}(x_{i+\frac{1}{2}}, y_j) + k_j \frac{h_i^- - h_{i-1}^+}{2} u_{2x}(x_{i-\frac{1}{2}}, y_j) \\
& - h_i \frac{k_{j+1}^- - k_j^+}{2} u_{2y}(x_i, y_{j+\frac{1}{2}}) + h_i \frac{k_j^- - k_{j-1}^+}{2} u_{2y}(x_i, y_{j-\frac{1}{2}}) \\
& + h_i \frac{{k_j^+}^2}{2} u_{2x,y}(x_i, y_j) - h_i \frac{{k_j^-}^2}{2} u_{2x,y}(x_i, y_{j-1}) + k_j \frac{{h_i^+}^2}{2} u_{x,2y}(x_i, y_j) - k_j \frac{{h_i^-}^2}{2} u_{x,2y}(x_{i-1}, y_j) \\
& - k_j \frac{{h_i^+}^2 - {h_i^-}^2}{2} p_x(x_i, y_j) u(x_i, y_j) - k_j \frac{{h_i^+}^2}{2} p_{ij} u_x(x_i, y_j) \\
& + k_j \frac{{h_i^-}^2}{2} p_{ij} u_x(x_{i-1}, y_j) - h_i \frac{{k_j^+}^2 - {k_j^-}^2}{2} p_y(x_i, y_j) u(x_i, y_j) \\
& - h_i \frac{{k_j^+}^2}{2} p_{ij} u_y(x_i, y_j) + h_i \frac{{k_j^-}^2}{2} p_{ij} u_y(x_i, y_{j-1}) \\
& - k_j (R^2_{i+\frac{1}{2},j} - R^2_{i-\frac{1}{2},j}) + \frac{{k_j^+}^2 - {k_j^-}^2}{2} (S^1_{ij} - S^2_{ij}) + T_{ij} \\
& - h_i (R^2_{i,j+\frac{1}{2}} - R^2_{i,j-\frac{1}{2}}) + \frac{{h_i^+}^2 - {h_i^-}^2}{2} (L^1_{ij} - L^2_{ij}) + H_{ij} \\
\text{(160)} \qquad & + U^1_{ij} + U^2_{ij} - N^1_{ij} + A_{ij} + B_{ij}
\end{aligned}$$

where

$$|A_{ij}| \leq c k_j {h_i^-}^2 h_{i-\frac{1}{2}} \|u\|_{2,\infty,\bar{\Omega}}, \tag{161}$$

$$|B_{ij}| \leq c h_i {k_j^-}^2 k_{j-\frac{1}{2}} \|u\|_{2,\infty,\bar{\Omega}}. \tag{162}$$

To simplify the expressions, let

$$b_{ij} = -k_j \frac{h_{i+1}^- - h_i^+}{2} v_{ij} \tag{163}$$

$$c_{ij} = -h_i \frac{h_{i+1}^- - h_i^+}{2} \left(-f(x_i, y_{j+\frac{1}{2}}) + p(x_i, y_{j+\frac{1}{2}}) u_{ij} - v_{ij}\right) \tag{164}$$

$$
\begin{aligned}
d_{ij} \quad &= h_i \frac{k_j^{+2}}{2} \partial_1^y v_{ij} - h_i \frac{k_j^{-2}}{2} \partial_1^y v_{i,j-1} + k_j \frac{h_i^{+2}}{2} \left(-f_x(x_i, y_j) + p_x(x_i, y_j) u_{ij} + p_{ij} \partial_1^x u_{ij} - \partial_1^x v_{ij}\right) \\
&- k_j \frac{h_i^{-2}}{2} \left(-f_x(x_{i-1}, y_j) + p_x(x_{i-1}, y_j) u_{i-1,j} + p_{i-1,j} \partial_1^x u_{i-1,j} - \partial_1^x v_{i-1,j}\right) \\
&- k_j \frac{h_i^{+2} - h_i^{-2}}{2} p_x(x_i, y_j) u_{ij} - k_j \frac{h_i^{+2}}{2} p_{ij} \partial_1^x u_{ij} + k_j \frac{h_i^{-2}}{2} p_{ij} \partial_1^x u_{i-1,j} \\
&- h_i \frac{k_j^{+2} - k_j^{-2}}{2} p_y(x_i, y_j) u_{ij} - h_i \frac{k_j^{+2}}{2} p_{ij} \partial_1^y u_{ij} + h_i \frac{k_j^{-2}}{2} p_{ij} \partial_1^y u_{i,j-1} \qquad (165)
\end{aligned}
$$

Now, the first correction $u_1^h = (u_{ij}^1)$ can be defined as follows

$$u_{0j}^1 = u_{M+1,j}^1 = u_{i0}^1 = u_{i,N+1}^1 = 0, \qquad (166)$$

and for $(i, j) \in \{1, ..., M\} \times \{1, ..., N\}$, we have

$$(\Delta^\tau u_1^h)_{ij} + h_i k_j p_{ij} u_{ij} = \int_{K_{ij}} f dx dy + b_{ij} - b_{i-1,j} + c_{ij} - c_{i,j-1} + d_{ij}. \qquad (167)$$

Using inequalities (89), (91), (92), (93), (96),(99)-(103), (142), (161) and (162) combined with the estimates obtained in lemma 3.2 yields the following $O(h)$ improvement

Theorem 3.4. *If the solution u of the equation (132) belonging to $C^4(\bar{\Omega})$ and $p \in C^2(\bar{\Omega})$. Let u^h be the basic finite volume solution of boundary condition (136) and discrete equation (137). Then the finite volume approximations $u_1^h = (u_{ij}^1)$ defined by the boundary condition (166) and the discrete equation (166), satisfies the following estimates*

$$\|e^1\|_{1,\tau} \leq ch^2 \|u\|_{4,\infty,\bar{\Omega}}, \qquad (168)$$

$$\left(\sum_{i,j} k_j h_{i+\frac{1}{2}} (e_{ij}^1)^2 \right)^{\frac{1}{2}} \leq ch^2 \|u\|_{4,\infty,\bar{\Omega}}, \qquad (169)$$

$$\|e^1\|_{L^2} \leq ch^2 \|u\|_{4,\infty,\bar{\Omega}}, \qquad (170)$$

where $e_{ij}^1 = u(x_i, y_j) - u_{ij}^1$ for $(i, j) \in \{1, ..., M\} \times \{1, ..., N\}$ and vanishes elsewhere.

4. Numerical Tests

4.1. **In one Dimensional Space.** In this subsection, we give two numerical tests justifying our theoretical results in one dimensional case. We mean by uniform mesh that so-called modified finite volume scheme and satisfying $h_{i+\frac{1}{2}} = h$, $h_{i+1}^- = h_i^+$ (see remark 2.5 in [7]) , and by cell-centered mesh that satisfying $h_i = \begin{cases} h, i \text{ is even,} \\ \frac{h}{2}, i \text{ is odd} \end{cases}$ and $h_i^- = h_i^+$. Note that, the first correction computed in case of uniform mesh is the second one, because $h_{i+1}^- - h_i^+ = 0$.
To show the convergence orders of the first correction and the basic finite volume solution, we compute the ratio

$$ratio = \frac{\log(e(h)) - \log(e(h_0))}{\log(h) - \log(h_0)}.$$

where h_0 is the initial value of h in each numerical test and $e(h)$ is the error corresponding to h. In the uniform mesh, we use the rule

$$ratio = -\frac{\log(e(1/2^{k+1})) - \log(e(1/2^k))}{\log 2}.$$

4.1.1. **First Test**. We consider the homogeneous equation $(\mathbf{I}) : -u_{xx} = f$ where $u(x) = sin(\pi x)$ and $f(x) = \pi^2 sin(\pi x)$.

TABLE 1. The convergence orders of the first correction and the basic solution in L^2-norm in uniform mesh.

h	correction		basic solution	
	order	error $/h^4$	order	error $/h^4$
1/32	-	0.0359	-	0.0298e+04
1/64	4.0004	0.0359	2.0003	0.1191e+04
1/128	4.0000	0.0359	2.0001	0.4764e+04
1/256	4.0032	0.0390	2.0000	1.9057e+04

TABLE 2. The convergence orders of the first correction and the basic solution in H_0^1-norm in uniform mesh.

h	correction		basic solution	
	order	error $/h^4$	order	error $/h^4$
1/32	-	0.1127	-	0.0935e+04
1/64	3.9999	0.1127	1.9999	0.3742e+04
1/128	3.9999	0.1129	2.0000	1.4967e+04
1/256	4.0032	0.1224	2.0000	5.9869e+04

TABLE 3. The convergence orders of the first correction and the basic solution in L^2-norm in cell-centered mesh.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
4/149	-	0.2474	-	0.3971
4/599	1.9903	0.2507	1.9986	0.3978
4/2999	1.9943	0.2516	1.9991	0.3981
4/14999	1.9961	0.2473	1.9994	0.3968

TABLE 4. The convergence orders of the first correction and the basic solution in H_0^1-norm in cell-centered mesh.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
4/149	-	0.5856	-	0.0281e+03
4/599	1.9929	0.5914	1.0000	0.1131e+03
4/2999	1.9958	0.5930	1.0000	0.5664e+03
4/14999	1.9971	0.5828	1.0000	2.8329e+03

TABLE 5. Comparaison between the accuracy of the first correction that uses first variant and the one using second variant in H_0^1 and L^2-norms in cell-centered mesh.

h	first variant		second variant	
	L^2-norm	H_0^1-norm	L^2-norm	H_0^1-norm
4/149	1.7827e-04	4.2204e-04	7.9569e-05	6.6012e-04
4/599	1.1180e-05	2.6374e-05	5.4358e-06	4.1591e-05
4/2999	4.4758e-07	1.0549e-06	2.2254e-07	1.6670e-06
4/14999	1.7591e-08	4.1450e-08	8.6276e-09	6.6477e-08

4.1.2. **Second Test**. In case of cell-centered mest, we saw that the convergence of the first correction is the same one of the basic solution in L^2-norm for the model (**I**). We present here an example of the mesh where the convergence order of the first correction improves really that one of the basic solution in both H_0^1 and L^2-norms for the model (**I**).
We consider $h_{i+\frac{1}{2}} = h$ and $x_{i+\frac{1}{2}} = \frac{2x_i+x_{i+1}}{3}$ for all $i = 1, ..., N-1$.

TABLE 6. The convergence orders of the first correction and the basic solution in L^2-norm .

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
$1/2^8$	-	0.3701	-	20.2232
$1/2^9$	1.9668	0.3788	1.0000	40.4472
$1/2^{10}$	1.9633	0.3832	1.0000	80.8948
$1/2^{11}$	1.9915	0.3855	1.0000	161.7899
$1/2^{12}$	2.0073	0.3835	1.0000	323.5799
$1/2^{13}$	1.9864	0.3871	1.0000	647.1598

TABLE 7. The convergence orders of the first correction and the basic solution in H_0^1-norm.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
$1/2^8$	-	1.4588	-	129.5822
$1/2^9$	2.1180	1.3443	0.9999	259.1881
$1/2^{10}$	2.0675	1.2828	1.0000	518.3882
$1/2^{11}$	2.0363	1.2509	1.0000	1.0368e+03
$1/2^{12}$	2.0301	1.2251	1.0000	2.0736e+03
$1/2^{13}$	1.9985	1.2263	1.0000	4.1471e+03

4.1.3. **Third Test**. We consider the homogeneous equation **(II)** : $-u_{xx} + u_x + u = f$ where $u(x) = sin(\pi x)$ and $f(x) = (\pi^2 + 1)\sin(\pi x) + \pi\cos(\pi x)$.

TABLE 8. The convergence orders of the first correction and the basic solution in L^2-norm in cell-centered mesh.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
2/75	-	0.1415	-	0.0099e+03
1/150	2.0074	0.1401	0.9796	0.0407e+03
1/750	2.0043	0.1398	0.9880	0.2051e+03
1/3750	2.0020	0.1402	0.9918	1.0270e+03

TABLE 9. The convergence orders of the first correction and the basic solution in H_0^1-norm in cell-centered mesh.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
2/75	-	0.4441	-	0.0377e+03
1/150	1.9936	0.4480	0.9887	0.1530e+03
1/750	1.9963	0.4491	0.9934	0.7681e+03
1/3750	1.9970	0.4502	0.9955	3.8438e+03

4.2. **In Two Dimensional Space.** In this subsection, we present two tests justifying our results of two dimensional space. The ratios are computed by using the first formula of ratio in subsection 5.1 .

4.2.1. **First Test**. We consider here $u = xy(1-x)(1-y)$, then u is the solution of (72), where $f = 2y(1-y) + 2x(1-x)$. The mesh considered here is such that $h_i = \begin{cases} h, i \text{ is even,} \\ \frac{h}{2}, i \text{ is odd} \end{cases}$ and $k_j = 3h/2$, with (x_i, y_j) is in the center of K_{ij}.

TABLE 10. Convergence orders of the first correction and basic solution in H_0^1-norm.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
0.02500	-	0.198028	-	2.51479
0.01667	1.99847	0.198151	1.02547	3.73344
0.01250	1.99906	0.198157	1.02262	4.95135
0.01000	1.99983	0.198139	1.02068	6.16896
0.00500	1.99992	0.198053	1.01593	12.2557
0.00333	2.00005	0.198007	1.01385	18.3418

TABLE 11. Convergence orders of the first correction and the basic solution in L^2-norm.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
0.02500	-	0.0439562	-	0.07314
0.01667	1.94335	0.0445737	1.99851	0.07318
0.01250	1.95651	0.0448951	1.99882	0.07320
0.01000	1.96233	0.045092	1.99900	0.07321
0.00500	1.97302	0.0454949	1.99935	0.07322
0.00333	1.97708	0.0456318	1.99947	0.07322

4.2.2. **Second Test**. In this test we choose $u(x,y) = \sin(\pi x)\sin(\pi y)$, then u is the solution of the problem (72), where $f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$. The mesh considered here is such that (x_i, y_j) is in the center of K_{ij}, $h_i = \begin{cases} h, i \text{ is even,} \\ \frac{h}{2}, i \text{ is odd} \end{cases}$ and $k_j = \begin{cases} h, j \text{ is even,} \\ \frac{h}{2}, j \text{ is odd} \end{cases}$.

TABLE 12. The convergence orders of the first correction and the basic solution in H_0^1-norm.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
0.033333	-	2.80465	-	34.946
0.016667	2.21288	2.4199	1.0016	69.815
0.008333	2.17818	2.1908	1.0010	139.59
0.004166	2.14753	2.06364	1.0007	279.16

TABLE 13. The convergence orders of the first correction and the basic solution in L^2-norm.

h	correction		basic solution	
	order	error $/h^2$	order	error $/h^2$
0.033333	-	0.14632	-	0.7812
0.016667	2.15317	0.131583	2.00058	0.7809
0.008333	2.05991	0.134658	2.00036	0.7802
0.004166	2.02539	0.138795	2.00025	0.7802

4.3. **Some Comments about the Numerical Results.**

(1) In **Table 1** and **Table 2**, numerical results show that on uniform mesh and for the model **(I)**, we can gain an $O(h^2)$- improvement in both H_0^1-norm and L^2-norm by the first correction.

(2) In **Table 3**, numerical results show that for the model **(I)**, we do not have an improvement in L^2-norm in the first correction when the mesh cell-centered. Furthermore, the coefficents of the error in the first correction are better than of those of the basic solution. To improve the order in L^2-norm, we compute the second correction.

(3) In **Table 4**, numerical results show that for the model **(I)**, we gain an $O(h)$-improvement by the first correction in H_0^1-norm.

(4) In **Table 5**, numerical results show that the accuracy of the error in the correction defined by second variant (see subsection 2.3.1) is batter than that of (17) in L^2-norm and contrary in H_0^1-norm.

(5) In **Table 6** and **Table 7**, numerical results show that the convergence of the first correction improves that of the basic solution in both H_0^1 and L^2 norms for **(I)**.

This implies that, on arbitrarily admissible mesh, the first correction improves the basic solution in L^2 and H_0^1 norms.

(6) In **Table 8** and **Table 9**, numerical results show that for the model **(II)**, we gain an $O(h)$-improvement in both L^2-norm and H_0^1-norm by the first correction for cell-centered mesh.

(7) In **Table 10** and **Table 12**, numerical results show that we gain an $O(h)$-improvement in H_0^1-norm by the first correction.

(8) In **Table 11** and **Table 13**, numerical results show that, the convergence order of the first correction is the same one as of the basic solution in L^2-norm, but the errors in the first correction are better than of those of the basic solution.

Remark 4.1. . *The idea used here is used by the authors to apply the defect correction technique in finite element method with non uniform mesh* [1]

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